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MARTINGALE PROOFS OF SOME GEOMETRICAL RESULTS IN BANACH SPACE THEORY

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Martingale techniques are used to give a new proof of the theorem of J. Bourgain-R. R. Phelps that a closed bounded convex subset K of a Banach space is the closed convex hull of its set of strongly exposed points provided K has the Radon-Nikodym property. The new notions of "ε-strong extreme points" and the approximate Krein-Milman property are introduced, and the intimate connections between these notions and "δ-trees" are explored. A self-contained treatment is given of the necessary martingale preliminaries, phrased in terms of quasi-martingales.

Our main objective is to render certain geometrical properties of convex sets accessible via the analytic techniques of martingale theory. For example, we give a new proof of the theorem of J. Bourgain [2] that a closed bounded convex subset K of a Banach space is the closed convex hull of its set of strongly exposed points provided K has the Radon-Nikodym property (the RNP). The result was first proved by R. R. Phelps assuming the entire Banach space has the RNP [9]. (Actually it has been observed by Larman and Phelps (see the remark following the proof of Theorem 4 of [8]) that a modification of Phelps' original argument can be used to obtain the above result of Bourgain). The arguments of Bourgain and Phelps seem to us to use ingenious but rather elaborate geometrical constructions. We obtain an essentially direct martingale proof via one geometrical result, Lemma 2.8. For the many equivalent formulations of the RNP, see [6]; we focus on the following one: K has the RNP if and only if every K-valued martingale converges almost everywhere. We also introduce the new notion of "ε-strong extreme points" and the approximate Krein-Milman property, and show the intimate connections between this notion and "δ-trees", that is, Banach valued dyadic martingales with differences everywhere δ-bounded away from zero.

Let us now indicate in greater detail the organization of this work. In the first section, we develop the needed properties of a slight generalization of martingales, termed quasi-martingales in the literature. Most of our results here are special cases of known results of Bellow [1] and Edgar and Sucheston [7]. In Theorem 1.1 we show that every quasi-martingale may be decomposed as the sum of a martingale and a sequence tending to zero in L^1 norm