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DIRECTLY FINITE ALEPH-NOUGHT-CONTINUOUS REGULAR RINGS

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This paper develops a structure theory for any directly finite, right $\aleph_0$-continuous, regular ring R, generalizing the corresponding theory for a right and left $\aleph_0$-continuous regular ring. The key result is that R must be unit-regular, which also provides a proof, much simpler than previous proofs, that any right and left $\aleph_0$-continuous regular ring is unit-regular. It is proved, for example, that R modulo any maximal two-sided ideal is a right self-injective ring. The Grothendieck group $K_0(R)$ is shown to be a monotone σ-complete interpolation group, which leads to an explicit representation of $K_0(R)$ as a group of affine continuous real-valued functions on the space of pseudo-rank functions on R. It follows, for example, that the isomorphism classes of finitely generated projective right R-modules are determined modulo the maximal two-sided ideals of R. For another example, if every simple artinian homomorphic image of R is a $t\times t$ matrix ring (for a fixed $t\in N$), then R is a $t\times t$ matrix ring.

Introduction. Recall that a regular ring R is said to be *right $\aleph_0$-continuous* provided the lattice $L(R_R)$ of principal right ideals of R is upper $\aleph_0$-continuous, i.e., every countable subset of $L(R_R)$ has a supremum in $L(R_R)$, and

$$A \wedge \left(\bigvee_{n=1}^{\infty} B_n \right) = \bigvee_{n=1}^{\infty} (A \wedge B_n)$$

for every $A \in L(R_R)$ and every countable ascending chain $B_1 \leqq B_2 \leqq \cdots$ in $L(R_R)$. Equivalently, R is right $\aleph_0$-continuous if and only if every countably generated right ideal of R is essential in a principal right ideal of R [1, Corollary 14.4]. Similarly, R is *left $\aleph_0$-continuous* provided the lattice $L(_RR)$ of principal left ideals of R is upper $\aleph_0$-continuous, or equivalently, the lattice $L(R_R)$ is lower $\aleph_0$-continuous. An *$\aleph_0$-continuous* regular ring is a regular ring that is both right and left $\aleph_0$-continuous.

The purpose of this paper is to show that the known structure theory for $\aleph_0$-continuous regular rings [3, 6, 7] carries over nearly intact for directly finite, right $\aleph_0$-continuous, regular rings. This may be viewed as a generalization, since all $\aleph_0$-continuous regular rings are directly finite [1, Proposition 14.20]. It is an open question whether all directly finite, right $\aleph_0$-continuous, regular rings are