PACIFIC JOURNAL OF MATHEMATICS
Vol. 101, No. 2, 1982

A NOTE ON THE GAUSS CURVATURE OF HARMONIC AND MINIMAL SURFACES

GIORGIO TALENTI

We present some inequalities for the Gauss curvature of embedded surfaces in euclidean 3-space, which are either graphs of harmonic functions or minimal. The proofs exploit the following facts: (i) a partial differential equation constrains the curvature of the surfaces in question; (ii) the differential equation constrains in a significant way, via an isoperimetric inequality, the level lines of the curvature.

1. Introduction. Let u be a (real-valued) harmonic function of two (real) variables x and y, and let K be the Gauss curvature of the graph of u. The following lemma is the starting point of our arguments.

LEMMA. *The following equation*

$$K(K_{xx} + K_{yy}) - K_x^2 - K_y^2 = 8K^3 \tag{1}$$

holds (*here subscripts stand for differentiation,* e.g., $K_x = \partial K/\partial x$, *etc.*).

Proof. The Gauss curvature of the graph of u is given by

$$K = (1 + u_x^2 + u_y^2)^{-2}(u_{xx}u_{yy} - u_{xy}^2) . \tag{2}$$

We have to show that formula (2) provides us with a kind of general solution of equation (1), i.e., formula (2) produces a solution to equation (1) whenever u is a harmonic function.

Indeed, our harmonic function can be represented (at least locally) as the real part

$$u(x, y) = \operatorname{Re} f(z) = \frac{1}{2}[f(z) + \overline{f(z)}] \tag{3}$$

of some holomorphic function f of the complex variable $z = x + iy$. Using (2) and (3) one easily finds the following alternative formula

$$K = -(1 + |f'|^2)^{-2}|f''|^2 , \tag{4}$$

where primes denote differentiation with respect to z.

The right-hand side of (4) involves a holomorphic function (namely f') and the first derivative of it. We want to eliminate this holomorphic function from (4) and equations involving deriva-