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# PTOLEMY'S INEQUALITY, CHORDAL METRIC, MULTIPLICATIVE METRIC

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**Ptolemy's inequality in $R^2$ states: If $A, B, C, D$ are vertices of a quadrilateral, then**

$$AB \cdot CD + BC \cdot AD \geqq AC \cdot BD$$

**with equality only $ABCD$ is a convex cyclic quadrilateral. A real normed linear vector space is called *ptolemaic* if**

$$\|x-y\|\,\|z\|+\|y-z\|\,\|x\| \geqq \|z-x\|\,\|y\|$$

**for all $x$, $y$ and $z$ in the space and it is called *symmetric* if**

$$\|\lambda x-y\|=\|x-\lambda y\|$$

**for all unit vectors $x$, $y$ and real $\lambda$. The equivalence of these two properties of a normed linear space is established and related results concerning distance functions in such spaces are proven.**

Although Ptolemy's inequality is a useful tool and has often been applied (e.g., see [7]) it does not seem to be as widely known as would be desirable. Recently Apostol [1] gave an elegant proof of this inequality using complex numbers in the plane (see also [2], [4] and [5]) and extended the inequality to $R^3$ thereafter. Apostol used Ptolemy's inequality to show that the chordal distance

$$\chi(a, b)=\frac{|a-b|}{\sqrt{1+|a|^2}\sqrt{1+|b|^2}},$$

defined for pairs of complex numbers, satisfies the triangle inequality $\chi(a, b)+\chi(b, c) \geqq \chi(a, c)$. In an earlier paper, Schoenberg [9], answering a problem raised by Blumenthal, proved the following: If $S$ is a real, seminormed space which is ptolemaic then the seminorm is a norm which springs from an inner product. In this note we wish to treat these results from a different point of view. We provide simpler proofs for some of the earlier results and extend a recent result of Schattschneider [6], [8].

2. DEFINITION 2. Let $X$ be real normed linear space with norm $\|\cdot\|$.

(i) $X$ is called *ptolemaic* if for every $x, y, z \in X$ we have

$$\|x-y\| \cdot \|z\|+\|y-z\| \cdot \|x\| \geqq \|x-z\| \cdot \|y\| . \tag{2.1}$$

(ii) $X$ is called *symmetric* if for every $x, y \in X$ with $\|x\|=$