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OPERATORS SIMILAR TO UNITARY OR SELFADJOINT ONES

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Let T be a bounded linear operator on a Hilbert space. General necessary and sufficient conditions are given in order that VTV^{-1} is unitary for some bounded linear operator V with bounded everywhere defined inverse. Similarly let B be a closed and densely defined linear operator in a Hilbert space. General necessary and sufficient conditions are given in order that VBV^{-1} is selfadjoint for some bounded linear operator V with bounded everywhere defined inverse.

1. Introduction and some preliminaries. Throughout this paper $\mathbf{H}$ is a complex Hilbert space with inner-product $(\ ,\ )$. Let A and B be linear operators with domain and range in $\mathbf{H}$. Then A and B are said to be *similar* if there exists a continuous linear operator V: $\mathbf{H} \to \mathbf{H}$ with bounded everywhere defined inverse such that $VA = BV$. Let T: $\mathbf{H} \to \mathbf{H}$ be a bounded linear operator. Then T is said to be *power bounded* if $\sup\{\|T^n\|: n \in \mathbf{N}\}$ is finite. Again let T: $\mathbf{H} \to \mathbf{H}$ be a continuous linear operator and suppose that its spectrum is contained in the circumference of the closed unit disc. The problem which poses itself is to find conditions on the resolvent family $\{(\lambda I - T)^{-1}: |\lambda| \neq 1\}$ which guarantee that T is similar to a unitary operator. Next let A be a closed linear operator with domain and range in $\mathbf{H}$. Suppose that its spectrum is a subset of $\mathbf{R}$. Find necessary and sufficient conditions on the resolvent family $\{(\lambda I - iA)^{-1}: \operatorname{Re}\lambda \neq 0\}$ in order that A is similar to a selfadjoint operator. The main tool we use is what might be called an operator valued Poisson kernel. If the spectrum of T is a subset of $\{\lambda \in \mathbf{C}: |\lambda| = 1\}$ and if T has inverse S, the corresponding Poisson kernel is given by

$$(1 - r^2)(I - re^{-i\theta}T)^{-1}(I - re^{i\theta}S)^{-1}, \qquad 0 \le r \le 1, -\pi \le \theta \le +\pi.$$

If A is a closed linear operator in $\mathbf{H}$ the spectrum of which is a subset of $\mathbf{R}$, then the corresponding Poisson kernel is given by

$$\omega\left(\omega^2 I + (\xi I - A)^2\right)^{-1} = \omega\left(((\omega + i\xi)I - iA)^{-1}((\omega - i\xi)I + iA)^{-1}\right), \qquad \omega > 0, \xi \in \mathbf{R}.$$

The present results generalize Theorems 1 and 2 in Van Casteren [**9**], where more related references can be found too. They are also closely related to a problem posed by Sz.-Nagy in [**3**, p. 585]. See also Sz.-Nagy and Foiaş [**8**, Chapitre IX, p. 334]. Another closely related paper is Stampfli [**5**]. This reference should also have been given in [**9**].