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THE JACOBSON DESCENT THEOREM

DAVID J. WINTER

A direct proof of the Jacobson Descent Theorem is given and used to prove the Jacobson-Bourbaki Correspondence Theorem.

The purpose of this paper is to give a proof of the *Jacobson Descent Theorem*, Theorem 1, which is direct in that it does not assume that $A = \mathrm{Hom}_{K^A} K$. This is then used to prove the *Jacobson-Bourbaki Correspondence Theorem*, Theorem 2. The approach simplifies earlier proofs.

A variation of a theme of Hochschild appearing in Jacobson [2] and Winter [3] recurs here in the concentrated form of the dual bases x_i, R_j which thread their way through both proofs. Thus, this paper underlines the importance of this natural duality.

Throughout the paper, K denotes a field, End K denotes the ring of endomorphisms of K as additive group, A denotes a subring of End K containing the K-span KI of the identity I of End K and V denotes a vector space over K of finite or infinite dimension $V: K$. Regard A as left K-vector space in the obvious way.

DEFINITION 1. An *A-product* on V is a mapping $A \times V \to V$, denoted $(T, v) \to T(v)$, such that V is an A-module and

$$(xT)(v) = x(T(v)) \quad (x \in K, T \in A, v \in V). \qquad \square$$

Clearly $T(v)$ $(T \in A, v \in K)$ is an A-product for K.

Suppose henceforth that $T(v)$ $(T \in A, v \in V)$ is an A-product for V, and $V^A = \{v \in V \mid T(xv) = T(x)v \text{ for } T \in A, x \in K\}$. In particular, we have then defined K^A.

DEFINITION 2. For k a subfield of K, a *k-form* of V is a k-subspace V' of V whose k-bases are K-bases of V. $\square$

THEOREM 1 (*Jacobson* [1]). *Let* $A: K < \infty$, *then* V^A *is a* K^A*-form of* V.

Proof. $\hat{K} = \{\hat{x} \mid x \in K\}$ separates A and therefore contains a basis $\hat{x}_1, \ldots, \hat{x}_n$ for the K-dual space $\mathrm{Hom}_K(A, K)$ of A where $\hat{x} \in \mathrm{Hom}_K(A, K)$ is defined for $x \in K$ by $\hat{x}(T) = T(x)$ $(T \in A)$. Letting $R_1, \ldots, R_n$ be a dual basis for A, so that $R_i(x_j) = \delta_{ij}$ $(1 \le i, j \le n)$, we have $T(xR_i)(x_j) = T(x\delta_{ij}) = T(x)\delta_{ij} = (T(x)R_i)(x_j)$ $(1 \le i, j \le n)$ so that $T(xR_i) = T(x)R_i$ $(1 \le i \le n)$ for all T, since the x_j separate A.