PACIFIC JOURNAL OF MATHEMATICS
Vol. 104, No. 2, 1983

BASIC CALCULUS OF VARIATIONS

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For the classical one-dimensional problem in the calculus of variations, a necessary condition that the integral be lower semicontinuous is that the integrand be convex as a function of the derivative. We shall see that, if the problem is properly posed, then this condition is also necessary for the k-dimensional problem. For the one-dimensional problem this condition is also sufficient. For the k-dimensional problem this condition is shown to be sufficient subject to an additional hypothesis. For the one-dimensional problem there is an existence theorem if the integrand grows sufficiently rapidly with respect to the derivative, and this result also holds for the k-dimensional problem, subject to an additional hypothesis. Some of these additional hypotheses are automatically satisfied for the one-dimensional problem.

Let G be a bounded domain in $\mathbf{R}^k$, $A = G \times \mathbf{R}^N$, Z be the space of $(N \times k)$-matrices and $F \in C(A \times Z)$. If $y: G \to \mathbf{R}^N$ is smooth, let $I_F(y) = \int_G F(x, y(x), y'(x))\,dx$ where $y'(x)$ is the matrix of partial derivatives of y.

If $k = N = 2$ and if $F(a, b, p) = |\det p|$ then I_F is the area integral which is lower semicontinuous though F is not convex in p for fixed (a, b). Thus the one-dimensional results do not, apparently, generalize.

There are $r = \binom{N+k}{k} - 1$ Jacobians of orders $1, \ldots, \min\{k, N\}$. Let $Y = \mathbf{R}^r$. There exists τ: $Z \to Y$ such that $\tau \circ y'(x) = J(y, x)$, where $J(y, x) = [J(y)](x)$, and $J(y)$ is the collection of all Jacobians of y, whenever y is a smooth map. If f: $A \times Y \to \mathbf{R}$ and if $f(\theta, \tau(p)) = F(\theta, p)$ for all (θ, p), then, evidently, $I(y) = I_F(y)$ where $I(y) = \int_G f(y_*(x), J(y, x))\,dx$ and $y_*(x) = (x, y(x))$.

If u: $V \times W \to X$ and if $v \in V$ let $u_v(w) = u(v, w)$ for each $w \in W$.

We define a class AC of transformations y for which each component of y and each component of $J(y)$, defined in a distribution sense, is in $L = L(G)$. We consider $I(y)$ to be the basic integral, not $I_F(y)$.

Let $T = \operatorname{range} \tau$. If $k = 1$ then $T = Y$ and T can be identified with Z so that $f = F$. In general, however, setting $f_\theta \circ \tau = F_\theta$ defines f_θ on $T \subset Y$ where $T \neq Y$. Let us say that f is *T-convex* if f_θ can be extended to a function which is convex over all of Y for each $\theta \in A$. Please notice that we do *not* require that f_θ be convex. What we do require is that there exist a convex function over all of Y which extends f_θ. Then a necessary condition that I be lower semicontinuous is that f be T-convex. If the extended function is also continuous over $A \times Y$, then the condition is also sufficient.

In some applications f, rather than F, may be given initially [**1**].