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COMPACT OPERATORS AND DERIVATIONS INDUCED BY WEIGHTED SHIFTS

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In this paper we study the question: which compact operators are contained in $\mathcal{R}(\delta_S)^-$, the norm closure of the range of the derivation $\delta_S(X) = SX - XS$ induced by a weighted shift S? We find that $\mathcal{R}(\delta_S)^-$ always contains the lower triangular (with respect to the basis (e_i) on which S is a shift) compact operators. Further, $\mathcal{R}(\delta_S)^-$ contains the n-lower triangular (operators T satisfying $(Te_i, e_j) = 0$ for $i - j > n$) compact operators if and only if $e_1 \otimes e_{n+1} \in \mathcal{R}(\delta_S)^-$. We also find necessary and sufficient conditions on the weights of S in order that $e_1 \otimes e_{n+1} \in \mathcal{R}(\delta_S)^-$ and that $\mathcal{K}$, the algebra of compact operators, be contained in $\mathcal{R}(\delta_S)^-$. These results completely answer the question: which essentially normal weighted shifts are d-symmetric?

Let $T \in \mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$. The derivation induced by T is the map $\delta_T(X) = TX - XT$ from $\mathcal{B}(\mathcal{H})$ to itself. Let $(e_n)_{n=1}^{\infty}$ (respectively $(e_n)_{n=-\infty}^{\infty}$) be an orthonormal basis for $\mathcal{H}$ and let S be the unilateral (respectively bilateral) weighted shift $Se_n = w_n e_{n+1}$, $n \in \mathbf{N}$ (respectively $n \in \mathbf{Z}$) with nonzero weights w_n. By taking a unitarily equivalent weighted shift, we may assume that $w_n = |w_n| > 0$.

Recall that for $f, g \in \mathcal{H}$, the operator $f \otimes g \in \mathcal{B}(\mathcal{H})$ is defined by $(f \otimes g)h = (h, g)f$ for $h \in \mathcal{H}$. In particular, $(e_i \otimes e_j)e_n = e_i$ if $n = j$ and $(e_i \otimes e_j)e_n = 0$ otherwise. In Theorem 2 we show that $e_1 \otimes e_{n+1} \in \mathcal{R}(\delta_S)^-$ if and only if $\Sigma_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{n+k-1} = \infty$. In Corollary 2, we find that this is also equivalent to $\mathcal{R}(\delta_S)^-$ containing all the n-lower triangular compact operators.

The above results enable us to characterize those essentially normal weighted shifts that are d-symmetric (i.e., satisfy $\mathcal{R}(\delta_S)^- = \mathcal{R}(\delta_S)^{-*}$). Namely, an essentially normal weighted shift is d-symmetric if and only if S satisfies the total products condition $\Sigma_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n} = \infty$ for all $n \in \mathbf{N}$. This yields another proof of the fact proved in Corollary 4 of [**8**] that all hyponormal (and hence all subnormal) weighted shifts are all d-symmetric.

THEOREM 1. *Let S be the unilateral (bilateral) weighted shift $Se_n = w_n e_{n+1}$ $n \in \mathbf{N}$ $(\mathbf{Z})$. Then $e_i \otimes e_j \in \mathcal{R}(\delta_S)$ for all $i, j \in \mathbf{N}$ $(\mathbf{Z})$ with $i > j$.*

Proof. Write $i = j + n$ with $n > 0$. Let $a_0 = 1/w_j$, $a_k = w_{j+n} \cdot \cdots \cdot w_{j+n+k-1}/w_j \cdot \cdots \cdot w_{j+k}$ for $k \geq 1$, and $a_k = 0$ for $k < 0$. Then