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SOME INEQUALITIES FOR PRODUCTS OF POWER SUMS

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We study the asymptotic behavior of the range of the ratio of products of power sums. For $x = (x_1, \ldots, x_n)$, define $M_p = M_p(x) = \Sigma x_i^p$. As two representative and explicit results, we show that the maximum and minimum of the function M_1M_3/M_2^2 are $\pm 3\sqrt{3}/16\, n^{1/2} + 5/8 + \mathcal{O}(n^{-1/2})$ and that $n \geq M_1M_3/M_4 > -n/8$, where "1/8" is the best possible constant. We give readily computable, if less explicit, formulas of this kind for $M_{p_1}^{a_1} \cdots M_{p_r}^{a_r}/M_q^b$, $\Sigma a_i p_i = bq$. Applications to integral inequalities are discussed. Our results generalize the classical Hölder and Jensen inequalities. All proofs are elementary.

1. Introduction and background. In this paper I shall discuss some inequalities involving power sums which build upon, and generalize, the Hölder and Jensen inequalities. Since the proofs, although elementary, involve lengthy and cumbersome computation, I shall indicate the main results and spirit of the paper in this introduction.

For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $p > 0$ define

$$M_p(x) = \sum_{i=1}^{n} x_i^p; \tag{1.1}$$

we exclude the possibility that some x_i is negative in (1.1) when p is not integral and set $M_0(x) \equiv n$.

MAIN THEOREM (*see* (3.5) *and* (3.17)). *Suppose*

$$f(x) = M_{p_1}^{a_1}(x) \cdots M_{p_r}^{a_r}(x)/M_q^b(x),$$

where $\Sigma a_i p_i = bq$ *and all parameters are positive. Let* M *denote the maximum value of* f *(*M *depends on* n*, the number of variables). Then there exist readily computable constants* c_i *so that* $M = c_1 n^{c_2} + \mathcal{O}(n^{c_3})$*. The minimum,* $\overline{m}$*, defined when all parameters are integers, in many cases satisfies* $\overline{m} = c_4 n^{c_2} + o(n^{c_2})$*, where* c_4 *is not always readily computable.*

Hölder's inequality (1.2) and Jensen's inequality (1.3) — see [**3**], p. 28 — state that for all x with $x_i \geq 0$ ($x \geq 0$),

$$M_p^a(x) M_r^c(x) \geq M_q^b(x) \text{ if } ap + cr = bq \text{ and } a, b, c, p, q, r \geq 0 \tag{1.2}$$