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APPLICATIONS OF DIFFERENTIATION OF $\mathfrak{L}_p$-FUNCTIONS TO SEMILATTICES

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Let S be a commutative semigroup with identity 1 such that $x^2 = x$ for each $x \in S$ (i.e. S is a semilattice). Let Γ denote the set of semicharacters equipped with topology of simple convergence and μ be a fixed probability measure on Γ. Those real-valued functions f on S which admit disintegrations of the form $f(x) = \int_\Gamma \rho(x)\, d\mu_f(\rho)$ where either $d\mu_f = f' d\mu$ with $f' \in L_p(\mu)$ $(1 \le p \le \infty)$ or μ_f is singular with respect to μ, are characterized. This extends the previous characterization of Alo and Korvin from the case where p is either 1 or ∞ to all $p \in [1, \infty]$. Applications of this theory to the classical L_p-spaces on the n-cube are also presented. The main applications occur upon specializing to the case where S is a Boolean algebra and the functions on S that are being disintegrated are additive. Not only is the Darst decomposition theorem easily recovered, but also the theory of V^P-spaces of set functions introduced by Bochner and extended by Leader is reproved from the point of view of "differentiation". As a by-product, it is shown that every non-atomic probability measure is in the closed convex hull (topology of simple convergence) of those zero-one-valued additive set functions which are not countably additive; a curious result when applied to Lebesgue measure.

1. Preliminary. For each $x \in S$, the shift operator E_x is defined on the class of all real-valued functions $f | S \to \mathbf{R}$ by $(E_x f)(y) = f(xy)$. Observe that $E_x E_y = E_{xy}$ and E_1 is the identity operator which we will also denote by I. We will be interested in certain difference operators of the form $\Delta = E_x \prod_{j=1}^{k}(I - E_{x_j})$ where $x, x_1, \ldots, x_k \in S$ and introduce the notation $\Delta f(yx; \{x_j\}) = (\Delta f)(y)$, at all times distinguishing between the function Δf and its evaluation $(\Delta f)(y)$, at y. It follows that $\Delta f(1)$ is the kth difference of f $(\Delta f(x; x_1, \ldots, x_k))$ as defined in [**6** and **8**]. Recall that a real-valued function f on S is called *completely monotonic* (CM) if $(\Delta f)(1) \ge 0$ for all choices of Δ. The class $\mathrm{CM}(S)$ of all completely monotonic functions is the same as the "positive definite functions" discussed in [**12**] and the difference operator Δ can be seen to be the operator "L" defined therein. Let $X = \{x_1, \ldots, x_k\}$ be a finite subset and Λ_X (Λ, when X is understood) denote the set of all σ_X (σ, when X is understood) of zero-one-valued functions on $\{1, 2, \ldots, k\}$ and let Δ_σ denote the difference operator $\prod_{j=1}^{k}(E_{x_j})^{\sigma_j}(I - E_{x_j})^{1-\sigma_j}$, where we adopt the convention that an operator (or member of any semigroup) to the power 0 is the identity even if that member is 0 itself. If f is a real-valued function on S then, following [**6**], we set $\| f \|_X = \sum_{\sigma \in \Lambda_X} | \Delta_\sigma f(1) |$. The triangle inequality implies $\| f \|_X$ is an increasing function of X (ordered by inclusion) and