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# SURJECTIVE EXTENSION OF THE REDUCTION OPERATOR

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**In this paper it is shown that there exists a Riemann surface $R$ and a nonnegative 2-form $P$ on $R$ such that the space of energy finite solutions of $d * du = uP$ on $R$ is properly contained in the space of Dirichlet finite solutions yet the subspaces of bounded functions in these two spaces coincide.**

Consider a nonnegative locally Hölder continuous 2-form $P$ on a hyperbolic Riemann surface $R$. Let $PX(R)$ denote the space of solutions of $d * du = uP$ on $R$ satisfying a certain boundedness property $X$, e.g. $D$ (finite Dirichlet integral $\int_R du \wedge *du$), $E$ (finite energy integral $\int_R du \wedge *du + u^2P$), $B$ (finite supremum norm) or the combinations $BD$ and $BE$. The reduction operator $T_X$ is defined to be the linear injection of the space $PX(R)$ into the space $HX(R)$ such that for each $u \in PX(R)$ there is a potential $p_u$ on $R$ with $|u - T_Xu| \leq p_u$. The unique existence of $T_X$ for the cases $X = B, D, E$ was established in [5] together with the representations

$$T_Xu(z) = u(z) + \frac{1}{2\pi}\int_R G_R(z, \zeta)u(\zeta)P(\zeta).$$

where $G_R(\cdot, \zeta)$ is the Green's function for $T$ with pole at $\zeta$.

One of the central questions concerning reduction operators is whether

(1) $\qquad T_{BX}$ is surjective implies that $T_X$ is surjective,

$X = D, E$. Since $PBX(R)$ is dense in $PX(R)$ in the same fashion as $HBD(R)$ is dense in $HD(R)$ (cf. [1], [4]), it is natural to conjecture that the implication (1) holds. Surprisingly, in [12] and [7] it was shown that (1) is false for $X = D, E$. Even the stronger conditions $\int_R P < +\infty$, $\int_{R\times R} G_R(z, \zeta)P(z)P(\zeta) < +\infty$ do not imply the surjectiveness of $T_E$ and $T_D$ respectively as was shown in [8], [9], [10].

In this connection we raise the question whether the fact that (1) does not hold for $X = E$ by itself implies that (1) does not hold for $X = D$. This is closely related to the following: Is it true that $PBD(R) = PBE(R)$ implies that $PD(R) = PE(R)$? We shall show here that the answer to the latter question is no even under the stronger assumption that $PBD(R) = PBE(R) \cong HBD(R)$ which is a consequence of the surjectiveness of $T_{BE}$. Therefore the former question will also be settled in the negative.