PACIFIC JOURNAL OF MATHEMATICS
Vol. 105, No. 1, 1983

LATTICE VERTEX POLYTOPES WITH INTERIOR LATTICE POINTS

DOUGLAS HENSLEY

Consider a convex polytope with lattice vertices and at least one interior lattice point. We prove that the number of boundary lattice points is bounded above by a function of the dimension and the number of interior lattice points. This extends to arbitrary dimension a result of Scott for the two dimensional case.

Introduction. In real Euclidean space $\mathbf{R}^D$ of dimension D there is the lattice $\mathbf{Z}^D$ of points with integer coordinates. Unless a different lattice is specified, a *lattice point* will mean a point of $\mathbf{Z}^D$, and a *lattice simplex* or *lattice convex polytope* will mean a simplex or convex polytope whose vertices are integer points, that is, elements of $\mathbf{Z}^D$. The interior in $\mathbf{R}^D$ of a set S is denoted by S°; if the affine span of S has dimension less than D, we denote the relative interior of S by S'.

Consider a lattice convex polytope $P \subseteq \mathbf{R}^D$ with the number $K = \sharp(P^\circ \cap \mathbf{Z}^D)$ of interior lattice points non-zero, and with a total of $J = \sharp(P \cap \mathbf{Z}^D)$ lattice points. *Our principal result is that J is bounded above by a function $B(K, D)$ of K and D alone.*

For the case of zero symmetric convex polytopes P there is no need to assume that the vertices are lattice points. By Van der Corput's generalization of Minkowski's theorem $\operatorname{vol}(P) \le K \cdot 2^D$ $[\mathbf{4}]^{40}$.† By a theorem of Blichfeldt, if the lattice points of P span $\mathbf{R}^D$, $J \le D + D!\operatorname{vol}(P)$ $[\mathbf{1}]^{55}$. Otherwise we can consider a subspace of $\mathbf{R}^D$ and get the same inequality $J \le D + D!K \cdot 2^D$. On the other hand if P need not be symmetric or have lattice point vertices then even for $D = 2$ and $K = 1$, J can be arbitrarily large. For instance, P might be the convex hull of $(-n, 0)$, $(0, 1 + 1/n^2)$, $(n, 0)$. With the restriction to lattice point vertices and $D = 2$ we have Scott's result that $J \le 3K + 7$ ($3K + 6$ for $K > 1$), and of course when $D = 1$ we have trivially $J \le K + 2$. These three bounds are best possible. Our results are far from best possible, but in any case the largest possible J grows rapidly with D, even for $K = 1$. Zaks, Perles and Wills have given examples of lattice simplices in $\mathbf{R}^D$ for which $K = 1$ and $J > 2^{2^{D-1}}$ [**11**]. There are some grounds for the belief that these examples are best possible. (See §4.) The existence of $B(K, D)$ will follow from some facts about Diophantine approximation which we now establish.

---

†Here the number above the brackets gives the page number on which this result is found in Lekkerkerker [7].