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ANALYTIC AND ARITHMETIC PROPERTIES OF THIN SETS

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Let $\mathbf{Z}^+$ and $\mathbf{Z}^-$ denote the sets of positive and negative integers respectively. We study relations between various thinness conditions on subsets E of $\mathbf{Z}^+$, with particular emphasis on those conditions that imply $\mathbf{Z}^- \cup E$ is a set of continuity. For instance, if E is a $\Lambda(1)$ set, a p-Sidon set (for some $p < 2$), or a UC-set, then E cannot contain parallelepipeds of arbitrarily large dimension, and it then follows that $\mathbf{Z}^- \cup E$ is a set of continuity; on the other hand there is a set E that is Rosenthal, strong Riesz, and Rajchman, which is not a set of continuity.

1. Introduction. Let $\mathbf{T}$ be the circle group and $\mathbf{Z}$ the integers; denote by $M(\mathbf{T})$ the customary convolution algebra of Borel measures on $\mathbf{T}$, and, given $\mu \in M(\mathbf{T})$ and $n \in \mathbf{Z}$, let

$$\hat{\mu}(n) = \int_{\mathbf{T}} e^{-in\theta}\, d\mu(\theta).$$

For a subset S of $\mathbf{Z}$ with infinite complement, call S a *set of continuity* if, for each number $\varepsilon > 0$ there is a $\delta > 0$ such that, if $\mu \in M(\mathbf{T})$, $\|\mu\| \leq 1$, then the condition that

$$(1.1) \qquad \limsup_{n \in \mathbf{Z} \setminus S} |\hat{\mu}(n)| < \delta \text{ implies that } \limsup_{n \in S} |\hat{\mu}(n)| < \varepsilon.$$

Less formally, S is a set of continuity if, for measures μ in the unit ball of $m(\mathbf{T})$, the size of $\limsup_{n \in S} |\hat{\mu}(n)|$ can be controlled by the size of $\limsup_{n \in \mathbf{Z} \setminus S} |\hat{\mu}(n)|$.

The definition of a set of continuity [**19**] was inspired by the theorem of K. de Leeuw and Y. Katznelson [**5**] to the effect that $\mathbf{Z}^+$ and $\mathbf{Z}^-$ are sets of continuity. Different proofs of the de Leeuw-Katznelson result were subsequently found by J. A. R. Holbrook [**22**] and L. Pigno [**34**]. Yet another proof will be presented in §2 of the present paper.

In §2, we consider analytic conditions. We first show that the theorem of de Leeuw and Katznelson is a rather direct consequence of Paley's theorem concerning the Fourier coefficients of $H^1(\mathbf{T})$-functions. We next recall the analytic conditions that E be a $\Lambda(1)$ set, a p-Sidon set ($p < 2$), or a UC-set, and we show by the same method that if E satisfies any of these conditions, then $Z^- \cup E$ is a set of continuity. We also exhibit an example of a set E which is a Rosenthal set, a strong Riesz set, and a Rajchman set, but is *not* a set of continuity; see §2 for all definitions.