PACIFIC JOURNAL OF MATHEMATICS
Vol. 105, No. 1, 1983

THE TWO-DIMENSIONAL DIOPHANTINE APPROXIMATION CONSTANT. II

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Given real numbers α and β, let $c_1(\alpha, \beta)$ denote the Diophantine approximation constant for the linear form $x + \alpha y + \beta z$ and let $c_2(\alpha, \beta)$ denote the corresponding dual constant for the simultaneous approximation of α and β. The paper gives various results about these constants in the case where α and β lie in some real cubic field. For example, it is shown that the suprema of $c_1(\alpha, \beta)$ and $c_2(\alpha, \beta)$, taken over all α, β such that $1, \alpha, \beta$ is an integral basis for a real cubic field, are equal, and a necessary and sufficient condition for this common value to be equal to $2/7$ is given.

1. Introduction. There is associated with each real number α a constant $c(\alpha)$ defined to be the infimum of those $c > 0$ such that the inequality

$$|x(\alpha x - y)| < c$$

has infinitely many solutions in integers x, y with $x \neq 0$. A well known theorem of Hurwitz states that $\sup c(\alpha)$, where the supremum is taken over all real numbers α, is equal to $1/\sqrt{5}$, and that $c(\alpha) = 1/\sqrt{5}$ only for certain numbers, such as $\frac{1}{2}(1 + \sqrt{5})$, in the algebraic extension $Q(\sqrt{5})$ of the rational field Q.

In the theory of simultaneous Diophantine approximation, there are two well known constants associated with each pair of real numbers α, β. One constant, which I denote by $c_1(\alpha, \beta)$, is defined to be the infimum of those $c > 0$ such that the inequality

$$|x + \alpha y + \beta z| \max(y^2, z^2) < c$$

has infinitely many solutions in integers x, y, z with y and z not both zero. The other constant, which I denote by $c_2(\alpha, \beta)$, is defined to be the infimum of those $c > 0$ such that the inequality

$$\max\left(|x|(\alpha x - y)^2, |x|(\beta x - z)^2\right) < c$$

has infinitely many solutions in integers x, y, z with $x \neq 0$.

Define $L = \sup c_1(\alpha, \beta)$ and $S = \sup c_2(\alpha, \beta)$, where the suprema are taken over all pairs of real numbers α, β. It is a well known unsolved problem to evaluate S. Cassels **[4]** showed that $S \geq 2/7$. Davenport **[11]** proved that $L = S$ and **[10]** that $S < .384$.

Define $L^* = \sup c_1(\alpha, \beta)$ and $S^* = \sup c_2(\alpha, \beta)$, where the suprema are taken over all α, β such that $1, \alpha, \beta$ is a basis of a real cubic field. It is