THE ROOT SUBGROUPS FOR MAXIMAL TORI
INFINITE GROUPS OF LIE TYPE

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Let $G_1$ be a finite group of Lie type defined over a field of characteristic $p$. The results of this paper represent an attempt to achieve a better understanding of the subgroup structure of $G_1$. It is somewhat surprising how limited our knowledge is, in this regard. For example, while centralizers of semisimple elements (i.e., $p'$-elements) of $G_1$ have been studied in detail and are fairly well understood, very little has been written about subgroups of $G_1$ generated by such centralizers. Even in explicit examples the analysis of such subgroups can be very difficult, the difficulty stemming from an inability to relate the generated group to the Lie structure of $G_1$. To deal with these situations and others we set up a framework that allows us to effectively study a fairly large class of subgroups of $G_1$ (those containing a maximal torus), by studying subgroups of the corresponding algebraic group. Essential to the development is a theory of root subgroups for arbitrary maximal tori of $G_1$.

1. Introduction. The theorems we establish have as their origin Lemma 3 of [22], which was later extended in [7] to show that if $q > 5$ and if $H$ is a Cartan subgroup of $G_1$ normalizing the $p'$-group $V$, then $V$ is the product of root subgroups of $H$. This result is quite useful and provided the starting point for the result in [21] which showed that with further field restrictions one could determine all $H$-invariant subgroups of $G_1$. For example, if $H \leq L \leq G_1$, then it was shown that $L$ could be generated by $N_L(H)$ together with certain of the root subgroups of $H$. Hence, $L$ is determined by a subset of the root system of $G_1$ together with a subgroup of the Weyl group of $G_1$. One wants to extend these results to cover the case of an arbitrary maximal torus, not just a Cartan subgroup. Therefore, one would like to develop a theory of root subgroups that makes sense for an arbitrary maximal torus and then establish results like those above. The present paper carries out this program.

The group $G_1$ satisfies $O^{p'}(\bar{G}_a) \leq G_1 \leq \bar{G}_a$, where $\bar{G}$ is a connected simple algebraic group over the closure of $F_\sigma$, and $\sigma$ is an endomorphism of $\bar{G}$ whose fixed point set, $\bar{G}_a$, is a finite group. Set $G = \bar{G}_a$ and $G_0 = O^{p'}(\bar{G}_a)$. A maximal torus of $G_1$ is a group of the form $T \cap G_1$, where $T = \bar{T}_a$ and $\bar{T}$ is a $\sigma$-invariant maximal torus of $\bar{G}$. The group $\bar{G}$ has a root system, $\Sigma$, and for each root $\alpha \in \Sigma$, there is a $\bar{T}$-root subgroup $U_\alpha$ of $\bar{G}$. These root subgroups are permuted by $\sigma$. If $\Delta$ is a $\langle \sigma \rangle$-orbit of such root subgroups, let $X = O^{p'}(\langle \Delta \rangle_\sigma)$, a subgroup of $G_1$. Such a group is