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H^∞ INTERPOLATION FROM A SUBSET OF THE BOUNDARY

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We obtain necessary and sufficient conditions for a bounded function on an open subset of the boundary of a smooth, bounded domain D in $\mathbf{C}^n$ to be the restriction of a holomorphic function from D into the unit disc. Our condition is a quadratic inequality involving the Szegö kernel of D which is the boundary analogue of the classical Pick-Nevanlinna condition for interpolation in the unit disc.

The classical interpolation theorem of Pick [**7**] and Nevanlinna [**5**] provides a necessary and sufficient condition for a function f defined on a subset $\{a_i\}$ of the open unit disc Δ to be the restriction of a holomorphic function from Δ into itself. This condition is a quadratic inequality involving the Szegö kernel of Δ. Specifically, an interpolating function of the required type exists if and only if for every finitely non-zero collection $\{\alpha_i\}$ of complex numbers we have

$$\sum S(a_i, a_j)\left(1 - f(a_i)\overline{f(a_j)}\right)\alpha_i\bar{\alpha}_j \geq 0, \tag{1}$$

where

$$S(z, \zeta) = \left[2\pi(1 - z\bar{\zeta})\right]^{-1}.$$

This result has been generalized by FitzGerald and Horn [**4**] (under the additional hypothesis that $\{a_i\}$ is a set of uniqueness for holomorphic functions) to more general sesquiholomorphic positive definite kernels defined in arbitrary domains in $\mathbf{C}^n$. In this note we address the analogous problem when the function f is specified on a subset of the boundary. In the case of the disc or the upper half plane, this problem has been considered by FitzGerald [**3**] and by Rosenblum and Rovnyak [**8**].

The method presented here applies to a rather general class of domains in $\mathbf{C}^n$. Our condition is similar to (1), but the summation must be replaced by integration. Thus, a bounded function f defined on an open subset E of the boundary (or distinguished boundary) of a domain D gives the boundary values of a holomorphic function from D into Δ if and only if for every $\rho \in L^2(E)$

$$\int_E\int_E S(z, \zeta)\left(1 - f(z)\overline{f(\zeta)}\right)\rho(z)\overline{\rho(\zeta)}\,d\sigma(z)\,d\sigma(\zeta) \geq 0. \tag{2}$$

Here $S(z, \zeta)$ is the Szegö kernel for D and σ is the induced measure on ∂D (or on the distinguished boundary in the case of a product domain).