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A NOTE ON THE CARDINALITY OF INFINITE PARTIALLY ORDERED SETS

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Let P be an infinite partially ordered set with 0 and 1. A subset B of P is called a π-base for P if for every element x of P with $0 < x < 1$ there exist elements b, c in B such that $0 < b \leq x \leq c < 1$. We let $\pi(P)$ denote the smallest cardinality of a π-base for P. We also let $h\pi(P) = \sup\{\pi(S) : S \subseteq P\}$. The width and depth of P are defined as usual: $w(P) = \sup\{\kappa : P$ contains an antichain of cardinality $\kappa\}$; $d(P) = \sup\{\kappa : P$ contains a well-ordered or dually well-ordered subset of cardinality $\kappa\}$. We establish the following result: THEOREM. $|P| \leq h\pi(P)^{d(P)}$. ***Various corollaries are obtained which imply and extend several known results on the cardinality of partially ordered sets, for example:*** COROLLARY. **(a)** $|P| \leq 2^{h\pi(P)}$. **(b)** $|P| \leq w(P)^{d(P)}$. **(c)** ***If B is a Boolean algebra then*** $|B| \leq 2^{w(B)}$.

1. **Preliminaries.** In this note several cardinality statements are established relating the cardinality of a partially ordered set to its width, depth, and π-weight. These results extend and imply several known results.

Our set-theoretic notation and terminology are standard and follow [3]. In particular the cardinality of a set S is denoted by $|S|$ and a cardinal number is thought of as an initial ordinal. If α and β are ordinals then α^β denotes the set of all mappings from β into α. If α and β are cardinals then we also let α^β denote the cardinal exponentiation of α to the power β. If κ is a cardinal then κ^+ denotes the first cardinal bigger than κ, and $\mathrm{cf}(\kappa)$ denotes the cofinality of κ—the least cardinal λ such that κ is the sum of λ cardinals each of which is less than κ.

For additional information on partially ordered sets and the concepts considered here the reader is referred to [5]. Let P be an infinite partially ordered set. The *width* of P, denoted by $w(P)$, is defined as

$$w(P) = \omega \cdot \sup\{\kappa : P \text{ contains an antichain of cardinality } \kappa\}.$$

We define

$$d^+(P) = \omega \cdot \sup\{\kappa : P \text{ contains a well-ordered subset of cardinality } \kappa\}$$

and

$$d^-(P) = \omega \cdot \sup\{\kappa : P \text{ contains a dually well-ordered subset of cardinality } \kappa\}.$$