PACIFIC JOURNAL OF MATHEMATICS
Vol. 107, No. 1, 1983

ON THE ZETA FUNCTION FOR FUNCTION FIELDS OVER F_p

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We consider here the zeta function for a function field defined over a finite field F_p. For each inter j, $\zeta(j)$ is a polynomial over F_p, as is $\zeta'(j)$, the "derivative" of zeta. In this note we compute the degree of these polynomials, determine when they are the constant polynomial and relate them to the polynomial gamma function.

In a recent series of papers D. Goss has introduced the notion of a zeta function $\zeta(j)$ for rational function fields over F_r, where $r = p^k$, with p a rational prime. In particular, for each positive integer i, with $i \not\equiv 0$ $(r-1)$, $\zeta(-i) \in F_r[t]$. Goss also defines the "derivative" of ζ, ζ', with $\zeta'(-i) \in F_r[t]$ if $i \equiv 0$ $(r-1)$. We combine these special values of ζ and ζ' into a single function $\beta(n)$ (with $n = -i$) defined by:

$$\beta(0) = 0, \qquad \beta(1) = 1, \tag{1}$$

$$\beta(n) = 1 - \sum_{\substack{i=1 \\ i \equiv n(s)}}^{n-1} \binom{n}{i} t^i \beta(i), \qquad n \geq 2,$$

where $s = r - 1$. Thus, by (3.9) and (3.10) of [2],

$$\beta(n) = \begin{cases} \zeta(-n), & n \not\equiv 0\ (s) \\ \zeta'(-n), & n \equiv 0\ (s) \end{cases}. \tag{2}$$

An important situation where these functions arise is in determining the class numbers of certain extension fields over $F_r[t]$ (modeled on cyclotomic fields). If P is a prime polynomial in $F_r[t]$, Goss defines class numbers $h^+(P)$ and $h^-(P)$ associated to P, in the classical fashion, and shows that their study (à la Kummer) involves the polynomials $\zeta(-i)$ and $\zeta'(-i)$. Thus it is important that we know certain facts about these functions, and hence about $\beta(n)$. Specifically, when is $\beta(n) = 1$? What is the degree of $\beta(n)$? When does $\beta(n)$ factor? In this note we give some answers to these questions, for the case $r = p$.

REMARK. I am indebted to Goss for bringing this material to my attention.