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ON THE UNIFORMIZATION OF CERTAIN CURVES

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The uniformization theorem of Poincaré and Koebe tells us that every smooth connected algebraic curve X over the complex numbers (or any Riemann surface) has as its universal covering space either the complex projective line $\mathbf{P}^1_{\mathbf{C}}$, the complex numbers $\mathbf{C}$, or the complex upper half plane $\mathfrak{H} = \{z \in \mathbf{C} \text{ s.t. } \operatorname{Im} z > 0\}$. When the universal covering space is the upper half plane $\mathfrak{H}$, we can regard the fundamental group $\pi_1(X)$ as a subgroup of $\mathrm{SL}_2(\mathbf{R})$ acting as covering transformations via linear fractional transformation. We shall focus on the case $\pi_1(X) \subset \mathrm{SL}_2(\mathbf{Z})$.

Introduction. Uniformization can be described by a class of differential equations on algebraic curves called Fuchsian equations (Poincaré [**6**] or Griffiths [**3**]). In this setting the monodromy representation of the differential equation gives us the inclusion of $\pi_1(X)$ into $\mathrm{SL}_2(\mathbf{R})$ and the solutions can be used to explicitly construct the covering map $\pi\colon \mathfrak{H} \to X$ much in the spirit of the Weierstrass $\wp$-function and the most classical case of uniformization, namely the inversion of the elliptic integral. Another classical case occurs when $X = \mathbf{P}^1_{\mathbf{C}} - \{0, 1, \infty\}$; which leads to the theory of the hypergeometric differential equation.

In this paper we will examine the specific case where X can be uniformized by a subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$, that is to say, the case where the map

$$\pi\colon \mathfrak{H} \to \mathfrak{H}/\Gamma \cong X$$

is the universal covering map. X will thus be a non-compact (because of the presence of cusps) modular curve. We will be particularly interested in giving an explicit description of which curves X arise this way. For example in the case of $\mathbf{P}^1_{\mathbf{C}}$-$\{n \text{ points}\}$ $n \geq 3$ we will give polynomial equations whose solutions represent those configurations of n-points whose complement can be uniformized by $\mathrm{SL}_2(\mathbf{Z})$, and the parameters necessary to construct the differential equation. The solutions to the equations will also be explicitly given.

The differential equations involved arise naturally in the theory of elliptic surfaces (Stiller [**7**]), but for the most part no use will be made of that fact.