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GAUTHIER'S LOCALIZATION THEOREM ON MEROMORPHIC UNIFORM APPROXIMATION

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This paper provides a proof of the localization theorem of Gauthier, which states that a function on a closed subset which is essentially of finite genus in an open Riemann surface is uniformly approximable by global meromorphic functions if and only if it is uniformly approximable locally by local meromorphic functions. The proof relies upon previously published work of Gauthier and of the author and these two lemmas: a connected surface of infinite genus cannot be the union of a compact set and a collection of pair-wise disjoint open sets of finite genus; if a Laurent series for an isolated essential singularity is prescribed at a point of a compact Riemann surface, there can be found an an analytic function on the surface with a singularity only at the given point and with Laurent series at the point identical with the given series except possibly for the coefficients of powers greater than $-2g$, where g is the genus of the surface.

For purposes of this paper a Riemann surface will be a connected one-dimensional complex manifold without boundary, and all functions are single-valued. An open surface is one which is not compact. A subset of a surface is called *bounded* if its closure is compact, and it is said to be *essentially of finite genus* if it is contained in a open set each connected component of which is of finite genus.

In **[G1]** Gauthier states in an equivalent way the following important localization theorem.

THEOREM 1. *Let M be an open Riemann surface, E a closed subset essentially of finite genus, and f a function defined on E with values in the extended complex plane. Suppose each point of E is interior to a compact disc D in M with the property that on $D \cap E$ f is the uniform limit of functions which are meromorphic on $D \cap E$. Then f is the uniform limit on E of meromorphic functions on M.*

Well-known examples **[GH]**, **[S2]** show that Theorem 1 is not valid in full generality without the hypothesis that E be essentially of finite genus. Gauthier pointed out to me **[G2]** that implicit in the proof of Theorem 1 is an additional hypothesis:

$$(*) \qquad M - E \text{ is not bounded.}$$