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CRAWLEY'S PROBLEM ON THE UNIQUE ω-ELONGATION OF p-GROUPS IS UNDECIDABLE

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Let G be an abelian p-group with $p^{\omega}G = 0$. Crawley has raised the following question: If all groups A with $p^{\omega}A$ cyclic of order p and $A/p^{\omega}A \cong G$ are mutually isomorphic, is G necessarily a direct sum of cyclic groups? We show this question to be independent of the axioms of set theory. Specifically, we prove that MA + $\neg$CH implies a negative answer for some G of cardinality $\aleph_1$; whereas, if $V = L$ is assumed, then every such G of cardinality $\aleph_1$ must be a direct sum of cyclic groups.

1. Introduction. All groups considered in this article are additively written p-primary abelian groups. If G is such a p-group, then $p^nG = \{p^nx : x \in G\}$ for $n < \omega$ and $p^{\omega}G = \bigcap_{n<\omega} p^nG$. We call G separable if $p^{\omega}G = 0$ and $\aleph_1$-separable if $p^{\omega}G = 0$ and every countable subset is contained in a countable direct summand. By a subsocle of G we mean a subgroup of the socle $G[p] = \{x \in G : px = 0\}$. We shall view the p-group G as a topological group endowed with the p-adic topology (the p^nG's form a neighborhood basis at 0) and its socle $G[p]$ as a topological vector space in the induced topology. A particularly prominent role in our considerations will be played by dense subsocles P of codimension one (that is, P is a dense subspace of $G[p]$ and $G[p]/P \cong Z(p)$, the cyclic group of order p). We shall write "Σ-cyclic" as an abbreviation for "a direct sum of cyclic subgroups."

It has been proved by Crawley **[2]** and by Hill and Megibben **[8]** that if the p-group G is Σ-cyclic, then any two p-groups A and B with $p^{\omega}A \cong p^{\omega}B$ and $A/p^{\omega}A \cong G \cong B/p^{\omega}B$ are necessarily isomorphic. That, conversely, Σ-cyclic groups are characterized as precisely the separable p-groups G satisfying this unique ω-elongation property was later established by Nunke **[12]** and Warfield **[14]**. But Crawly had previously raised the question of a somewhat stronger converse: If G is a separable p-group with the property that all groups A with $p^{\omega}A \cong Z(p)$ and $A/p^{\omega}A \cong G$ are mutually isomorphic, is G necessarily Σ-cyclic? We find it convenient to use the term "Crawley group" for such p-groups G. The conjecture that the Crawley groups are precisely the Σ-cyclic p-groups appears, in view of the Nunke-Warfield theorem, quite promising; and, assuming CH, Warfield **[14]** succeeded in showing that every Crawley group with countable basic subgroup is in fact Σ-cyclic. On the otherhand, the Nunke-Warfield result strongly uses the fact that $p^{\omega}A$ is allowed to be uncountable and, mindful of the analogous impact that countability considerations have on