PACIFIC JOURNAL OF MATHEMATICS
Vol. 107, No. 1, 1983

SUFFICIENCY AND RELATIVE ENTROPY IN $*$-ALGEBRAS WITH APPLICATIONS IN QUANTUM SYSTEMS

FUMIO HIAI, MASANORI OHYA AND MAKOTO TSUKADA

The sufficiency and weak sufficiency in $*$-algebras are discussed. Some properties are studied concerning the relative entropy and the sufficiency for invariant states and KMS states in W^*- and C^*-dynamical systems.

Introduction. The concept of sufficiency is very important in mathematical statistics. The abstract measure theoretic investigation of sufficient statistics was initiated by Halmos and Savage [**13**]. Kullback and Leibler [**19**] gave the characterization of sufficiency in terms of the information (i.e., the classical relative entropy). Umegaki [**33, 34**] studied the sufficiency and the relative entropy in the noncommutative case of semi-finite von Neumann algebras.

Araki [**4, 5**] extended the relative entropy to the case for normal positive linear functionals of general von Neumann algebras and showed its several properties. Furthermore Uhlmann [**32**] showed the general WYDL concavity using a quadratic interpolation theory and defined the relative entropy of positive linear functionals of arbitrary $*$-algebras.

In the previous paper [**14**], we discussed the sufficiency and the relative entropy in von Neumann algebras and gave the characterizations of invariant states and KMS states with respect to the modular automorphism group of a faithful normal state.

In this paper, we further develop the sufficiency and the relative entropy in $*$-algebras. In §1, we introduce besides the sufficiency another notion of weak sufficiency and establish the relation between them. In §2, we deal with the weak sufficiency of positive linear maps between $*$-algebras. In §3, we mention the Araki's and Uhlmann's relative entropies which are equal in the von Neumann algebra case. We further give a formula of relative entropy for states of C^*-algebras. In §4, we establish some properties of invariant states and KMS states in W^*-dynamical systems and C^*-dynamical systems through the relative entropy and the sufficiency. The theorems there improve or extend the results obtained in [**14**]. Finally we give an application to the Gibbs states of quantum lattice systems.