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SOME POINCARÉ SERIES RELATED TO IDENTITIES OF 2 × 2 MATRICES

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A partial solution to a problem of Procesi has recently been given by Formanek, Halpin, Li by determining the Poincaré series of the ideal of two variable identities of $M_2(k)$. Two related results are obtained in this article.

A weak identity of $M_n(k)$ is a polynomial which vanishes identically on sl_n, the subspace of $M_n(k)$ of matrices of trace zero. We show that the Poincaré series of the ideal of two variable weak identities of $M_2(k)$ is rational. In addition it is shown that the ideal of identities of upper triangular 2 × 2 matrices in an arbitrary finite number of variables has a rational Poincaré series. As an application we are able to determine this ideal precisely.

Introduction. Let $S = K\langle x_1,\ldots,x_n\rangle$ be the free associative algebra over k where k is any field of characteristic zero. S is naturally graded by giving x_1 degree $(1,0,\ldots,0)$, x_2 degree $(0,1,\ldots,0)$, etc. Denote by $S_{(i_1,\ldots,i_n)}$ the subspace of S generated by monomials of degree $(i_1,\ldots,i_n)$. If A is a homogeneously generated ideal of S then we associate a series to A, called the Poincaré series of A, via

$$P(A) = \sum_{i_1,\ldots,i_n \geq 0} a(i_1,\ldots,i_n) s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n}$$

where $a(i_1,\ldots,i_n) = \dim_k(A \cap S_{(i_1,\ldots,i_n)})$. In **[1]** Formanek, Halpin, Li showed that the Poincaré series of the ideal of two variables identities of $M_2(k)$ is a rational function in s_1 and s_2. In this article we obtain two related results.

A weak identity of $M_n(k)$ is a polynomial which vanishes upon substitution of elements of $\mathrm{sl}_n(k)$, where $\mathrm{sl}_n(k)$ denotes the subspace of $M_n(k)$ of matrices of trace zero. The notion of a weak identity was introduced by Razmyslov **[2]** in connection with the study of central polynomials. Let $T_2^W(x_1, x_2)$ denote the ideal of $k\langle x_1, x_2\rangle$ of weak identities of $M_2(k)$. In Section 1 we determine $P(T_2^W(x_1, x_2))$ and find that it is again a rational function in s_1 and s_2.

In §2 we consider the identities of the subalgebra of $M_2(k)$ consisting of upper triangular matrices. By restricting to upper triangular matrices we are able to obtain results more complete than those obtained in **[1]**. We