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A NOTE ON TAMELY RAMIFIED EXTENSIONS OF RINGS

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Buhler gave a criterion for a class of finite free extensions of discrete valuation rings to be tamely ramified 1-dimensional regular rings. In this note, we extend this criterion to finite free extensions of general local rings and, in the final section, indicate the extension to schemes.

1. Introduction. To set the notation, let A be a noetherian local ring of Krull dimension n and let $A \to B$ be a finite free extension of rings; denote by $\mathfrak{d}_{B/A}$ the discriminant of this extension, defined as $\det[\operatorname{tr}(b_i b_j)]$ where $b_1,\ldots,b_m$ is a free basis of B over A and tr: $B \to A$ denotes the trace morphism. Let $\mathfrak{m}_A$ be the maximal ideal of A and define a function $\nu_{\mathfrak{m}_A}$ on A by $\nu_{\mathfrak{m}_A}(x) = r$ where r is the largest integer with $x \in \mathfrak{m}_A^r$ and $\nu_{\mathfrak{m}_A}(0) = \infty$. Note that $\nu_{\mathfrak{m}_A}$ is a valuation if $\operatorname{gr}_A(\mathfrak{m}_A)$ has no zero divisors, in particular if A is regular **[2]**.

If $\mathfrak{n}_1,\ldots,\mathfrak{n}_s$ are the maximal ideals of B lying over $\mathfrak{m}_A$ define the *ramification index* $e_{\mathfrak{n}_i/\mathfrak{m}_A}$ to be $l_{B_{\mathfrak{n}_i}}(B_{\mathfrak{n}_i}/\mathfrak{m}_A B_{\mathfrak{n}_i})$ where $l_B(M)$ denotes the length (of a composition series) of the artin B-module M. If A is a discrete valuation ring, the $e_{\mathfrak{n}_i/\mathfrak{m}_A}$ clearly coincide with the usual ramification indices of algebraic number theory. Recall that the embedding dimension $\operatorname{ed}(B)$ of the semi-local ring B is $\max \dim_{\kappa(\mathfrak{n}_i)} \mathfrak{n}_i/\mathfrak{n}_i^2$ where $\mathfrak{n}_i$ runs through all maximal ideals of B. With the above notation the main result of this paper is:

THEOREM 1. *If A is regular* (resp. $\operatorname{gr}_A(\mathfrak{m}_A)$ *has no zero divisors*) *and if $B = A[X]/\langle f(X)\rangle$ where $f(X)$ is a monic polynomial and $\kappa(\mathfrak{m}) \to \kappa(\mathfrak{n}_i)$ is separable for all $i = 1,\ldots,s$, then*

$$\nu_{\mathfrak{m}_A}(\mathfrak{d}_{B/A}) \geq \sum_{i=1}^{s} \left(e_{\mathfrak{n}_i/\mathfrak{m}_A} - 1\right)[\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m}_A)]$$

with equality if and only if (resp. *only if*) $\operatorname{ed}(B) = \operatorname{ed}(A)$ *and B is tamely ramified over A in that $p \nmid e_{\mathfrak{n}_i/\mathfrak{m}_A}$ for all i, where p is the characteristic of $\kappa(\mathfrak{m}_A)$.*