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LOCALIZATIONS OF TORSION THEORIES

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If R is an h-local domain, then the hereditary torsion theories of R are described in terms of the hereditary torsion theories of R_M for all maximal ideals M of R. By means of an example, it is shown that h-local is too strong a hypothesis for this localization property. As an application, all the hereditary torsion theories of h-local Prüfer domains are described. Some equivalent conditions for a domain to be h-local are generalized to conditions about hereditary torsion theories.

Introduction. R will always denote an integral domain and all R-modules are unital modules. spec R will denote the set of all prime ideals of R and mspec R will denote the set of all maximal ideals of R. The main purpose of this paper is to describe the hereditary torsion theories of R in terms of the hereditary torsion theories of the localizations R_M for all $M \in$ mspec R. We will generally follow the terminology of the text by B. Stenström [**9**]. Because of the bijective correspondence between the hereditary torsion theories of R and the Gabriel topologies of R [**9**, Ch. 6, Theorem 5.1], it suffices to describe the Gabriel topologies of R. The results will be described mostly in terms of the Gabriel topologies of R.

We remind the reader of the definition. A *Gabriel topology of* R is a non-empty family $\mathfrak{F}$ of ideals of R satisfying axioms T1-T4:

T1. If $I \in \mathfrak{F}$ and $I \subset J$ for J an ideal of R, then $J \in \mathfrak{F}$.

T2. If $I, J \in \mathfrak{F}$, then $IJ \in \mathfrak{F}$.

T3. If $I \in \mathfrak{F}$ and $r \in R$, then $(I : r) \in \mathfrak{F}$.

T4. If I is an ideal of R and $J \in \mathfrak{F}$ with $(I : r) \in \mathfrak{F}$ for all $r \in J$, then $I \in \mathfrak{F}$.

The condition that $\mathfrak{F}$ is non-empty is equivalent to requiring $R \in \mathfrak{F}$. Note that condition T2 has been changed from the condition "if $I, J \in \mathfrak{F}$, then $I \cap J \in \mathfrak{F}$" of the Stenström text [**9**] to the present equivalent form for commutative rings. It is easily seen that T3 and T4 imply T1 and T2, and since the rings considered in this paper are all commutative, T1 implies T3. Thus to show that $\mathfrak{F}$ is a Gabriel topology of R it suffices to verify T3 and T4, or to verify T1 and T4. It follows immediately from T2 that if $I \in \mathfrak{F}$, then $I^n \in \mathfrak{F}$ for all $n \geq 0$. Given a Gabriel topology $\mathfrak{F}$ of R, the class of torsion R-modules of the corresponding hereditary torsion theory consists of all R-modules T such that $\operatorname{Ann}_R(x) \in \mathfrak{F}$ for all $x \in T$.