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# POWERS OF IDEALS IN LOCALLY UNMIXED NOETHERIAN RINGS

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**It is known that the following statements are equivalent for a semi-local ring $R$: (1) $R$ is analytically unramified; (2) There exists an open ideal $I$ in $R$ and an integer $n \geq 0$ such that $(I^{n+i})_a \subseteq I^i$ for all $i \geq 1$, where $(I^{n+i})_a$ is the integral closure of $I^{n+i}$. Moreover, if $R$ is analytically unramified and $I$ is any ideal in $R$, then (2) holds for $I$ and, (3) There exists an integer $m \geq 1$ such that, with $B = (I^m)_a$, $(B^i)_a = B^i$ for all $i \geq 1$. The main result in this paper shows that an analogous theorem holds with reduced unmixed local ring and $I^{[i]}$ replacing analytically unramified semi-local ring and $(I^i)_a$, respectively, where $I^{[i]}$ is the intersection of certain primary ideals related to $I^i$. An application and a generalization are included.**

**1. Introduction.** The theorem on analytically unramified semi-local rings in the abstract can be found in [**15**, Theorem 2], where it is actually shown that (1) and (2) are each equivalent to (3) holding for some open ideal $B$ $(= (I^m)_a)$. And the equivalence of (1) and (2) is also given in [**14**, Lemma 1 and Theorem 1.4]. I have always thought this theorem was rather beautiful, and it is also quite useful. For example, D. Rees developed it and then used it in [**14**] to characterize analytically unramified local rings as local rings $R$ all of whose finitely generated overrings have finite integral closures. (An overring of $R$ is a ring containing $R$ and contained in the total quotient ring of $R$.) This result verified a (modified) conjecture of O. Zariski, [**16**].

There has recently been some renewed interest in large powers of ideals and their integral closures, and in some of the new results in this area the above theorem and the closely related Valuation Theorem of Rees, [**12**], have proved quite useful. (For example, see [**10**] and [**11**].) One of the main results in this paper, (13), shows that a similar and closely related theorem holds for ideals in reduced unmixed local rings. The ideals $I^{[i]}$ in the theorem are not as easily described as are the ideals $(I^i)_a$, but they are well defined ideals, and the theorem implies that if $I$ is any ideal in such a ring $R$ and $I^{\langle i \rangle} = \bigcap \{I^i R_P \cap R;\ P \text{ is a prime divisor of } (I^i)_a\}$, then there exists an integer $n \geq 0$ such that $I^{\langle n+i \rangle} \subseteq I^i$ for all $i \geq 1$. (See (17).) For ideals $I$ of the principal class in reduced Cohen-Macaulay rings these theorems were previously known, since then $I^i = I^{[i]} = I^{\langle i \rangle}$ (see