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REGULAR EMBEDDINGS OF A GRAPH

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In this paper we study embeddings of a graph G in Euclidean space R^n that are 'regular' in the following sense: given any two distinct vertices u and v of G, the distance between the corresponding points in R^n equals α if u and v are adjacent, and equals β otherwise. It is shown that for any given value of $s = (\beta^2 - \alpha^2)/\beta^2$, the minimum dimension of a Euclidean space in which G is regularly embeddable is determined by the characteristic polynomials of G and $\bar{G}$.

1. Introduction. To embed a graph in Euclidean spaces with various restrictions, and to find the minimum dimension of the space for these embeddings, are interesting problems [**1**], [**4**], [**5**]. In this paper we consider a regular embedding of a graph.

An embedding of a graph G in a Euclidean space R^n is called a *regular embedding* of G provided that, for any two distinct vertices u and v of G, the distance between the corresponding points in R^n equals α if u and v are adjacent, and equals β otherwise. The vertices of G are mapped onto distinct points of R^n, but there is no restriction on the crossing of edges. The value $s = (\beta^2 - \alpha^2)/\beta^2$ is called the *parameter* of the regular embedding. Let $\dim(G, s)$ denote the minimum number n such that G can be regularly embedded in R^n with parameter s.

Consider, for example, the circuit graph C_5. For every regular embedding of C_5, it is seen that

$$\tfrac{1}{2}\left(-\sqrt{5} - 1\right) \leq s \leq \tfrac{1}{2}\left(\sqrt{5} - 1\right)$$

and

$$\dim(C_5, s) = \begin{cases} 2 & \text{if } s = \frac{1}{2}\left(\pm\sqrt{5} - 1\right), \\ 4 & \text{otherwise.} \end{cases}$$

The 'critical' embeddings of C_5 in R^2 with $s = \frac{1}{2}(\pm\sqrt{5} - 1)$ are illustrated in Fig. 1.

Let $\phi(G; x)$ denote the characteristic polynomial of a graph G (that is, $\phi(G; x) = |x\mathbf{I} - \mathbf{A}(G)|$), and put

$$\Phi(G; x) = \phi(G; -x) - (-1)^g \phi(\bar{G}; x - 1),$$