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ON A RADON-NIKODYM PROBLEM FOR VECTOR-VALUED MEASURES

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The purpose of this paper is to show that if m is a Banach space-valued measure with finite variation on a σ-algebra, then the variation $|m|$ of m has a Radon-Nikodym derivative with respect to m.

This Radon-Nikodym derivative takes its values in the dual of the Banach space, is integrable in Dinculeanu's sense and may be chosen of norm as close to one as we want.

From this theorem we deduce that if m and m' are Banach space-valued measures on the same σ-algebra, such that $m \ll m'$, then m has a Radon-Nikodym derivative with respect to m' and this derivative is m'-integrable in Dinculeanu's sense if we assume that the image space of m has Radon-Nikodym property.

1. Introduction. The general setting is the following. T will denote a set, $\mathfrak{A}$ a σ-algebra of subsets of T, E a Banach space with dual E' and m a measure from $\mathfrak{A}$ to E, with finite variation.

We will show that for every $\varepsilon > 0$, there exists a function f from T to E' which is strongly measurable, integrable with respect to m in Dinculeanu's sense, such that $|m|(A) = \int_A f\,dm$ for every A in $\mathfrak{A}$ and $1 \leq |f| < 1 + \varepsilon$.

Let us now recall the different ways to define integrable functions with respect to an operator-valued measure. Let E and F be Banach spaces, m a measure from $\mathfrak{A}$ to $\mathfrak{B}(E, F)$ and f a measurable function from T to E.

(a) f is integrable in Dobrakov's sense if there exists a sequence $(f_n)_{n\geq 1}$ of step functions, converging a.e. to f, such that for every A in $\mathfrak{A}$, the sequence $(\int_A f_n\,dm)_{n\geq 1}$ is convergent in F. The limit of this sequence is then denoted by $\int_A f\,dm$.

(b) f is integrable in the author's sense if there exists a sequence $(f_n)_{n\geq 1}$ of step functions, converging a.e. to f, such that

$$\lim_{n,p\to\infty} \int |f_n - f_p|\,d|x' \circ m| = 0$$

uniformly in x' in E', $\|x'\| \leq 1$.

(c) f is integrable in Dinculeanu's sense if there exists a sequence $(f_n)_{n\geq 1}$ of step functions, converging a.e. to f, such that

$$\lim_{n,p\to\infty} \int |f_n - f_p|\,d|m| = 0$$

i.e. the function $|f|$ is $|m|$-integrable.