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ON THE BEHAVIOR NEAR A TORUS OF FUNCTIONS HOLOMORPHIC IN THE BALL

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If f is bounded and holomorphic in the unit ball in $\mathbf{C}^n$ then it has radial limits at almost all points of the boundary of the ball. More is true; for example, f will have limits almost everywhere with respect to arclength on any arc that forms part of the boundary of an anlaytic disc. Motivated by these considerations we consider an n-dimensional torus in the boundary of the ball and ask if there are growth conditions less restrictive than boundedness that imply the existence of radial limits on this torus. We show that the answer is no for some of the standard function classes. For example, we show that there is holomorphic function of bounded mean oscillation in the ball that has a finite radial limit at no point of the torus.

Let B_n denote the unit ball in $\mathbf{C}^n$ and let σ_n be Lebesgue measure on its boundary, ∂B_n, normalized so that $\sigma_n(\partial B_n) = 1$. If f is a holomorphic in B_n, we say that $f \in H^p(B_n)$, $0 < p < \infty$, if

$$\| f \|_p^p = \sup_{0<r<1} \int_{\partial B_n} | f(r\xi) |^p \, d\sigma_n(\xi) < \infty;$$

we say $f \in H^\infty(B_n)$ if $\| f \|_\infty = \sup_{\xi \in B_n} | f(\xi) | < \infty$. If $f \in H^2(B_n)$ we say that $f \in \mathrm{BMO}(B_n)$ if $\exists$ a constant C such that for all $F \in H^2(B_n)$ we have $| \int_{\partial B_n} F\bar{f} \, d\sigma_n | \leq C \| F \|_1$. Then $\mathrm{BMO}(B_n)$ serves as the dual of $H^1(B_n)$ and we have $H^\infty(B_n) \subseteq \mathrm{BMO}(B_n) \subseteq H^p(B_n)$, $0 < p < \infty$. For a more intrinsic description $\mathrm{BMO}(B_n)$, see [1].

Next we describe some function spaces in the open unit disc U in the complex plane. If μ is a positive measure on U then $A^p(d\mu)$ will denote the space of holomorphic functions in $L^p(d\mu)$, $0 < p \leq \infty$. When $d\mu(r, \theta) = (1 - r)^\alpha \, dr \, d\theta$, $\alpha > -1$, we use the notation $A^p(d\mu) = A^p_\alpha$. Finally we say that g is a Bloch function, $g \in \mathfrak{B}(U)$, if

$$\| g \|_{\mathfrak{B}} = \sup_{|z|<1} (1 - | z |) \, | g'(z) | < \infty.$$

We have a mapping π: $\mathbf{C}^n \to \mathbf{C}$ given by $\pi(z_1, \ldots, z_n) = n^{n/2} \prod_{j=1}^n z_j$. It is easily checked that $\pi(B_n) = U$, $\pi(\bar{B}_n) = \bar{U}$, and that $\pi^{-1}(\partial U) = T_n = \{(z_1, \ldots, z_n) \colon | z_j | = n^{-1/2}, j = 1, \ldots, n\}$. In this paper it is shown that if $g \in A^p_{(n-3)/2}$ then $g \circ \pi \in H^p(B_n)$, and that if $g \in \mathfrak{B}(U)$ then $g \circ \pi \in \mathrm{BMO}(B_n)$.