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A MINIMAL UPPER BOUND ON A SEQUENCE OF TURING DEGREES WHICH REPRESENTS THAT SEQUENCE

HAROLD T. HODES

Given a sequence of Turing degrees $\langle a_i \rangle_{i<\omega}$, $a_i < a_{i+1}$, is there a function of f such that (i) $\deg(f)$ is a minimal upper bound on $\langle a_i \rangle_{i<\omega}$, and (ii) $\{\deg((f)_n) \mid n < \omega\} = \{a_i \mid i < \omega\}$? In this note we show that the most natural minimal upper bound on $\langle a_i \rangle_{i<\omega}$ is of the form $\deg(f)$ for such an f.

Because there seem to be a cluster of interesting notions and question related to this problem, we start with some definitions. Fix a recursive pairing function $(x, y) \mapsto \langle x, y \rangle$; $(f)_x(y) = f(\langle x, y \rangle)$. Where I is a set of Turing degrees and $f \in {}^\omega\omega$, f represents (subrepresents) I iff $I = \{\deg((f)_n) \mid n < \omega\}$ ($I \subseteq \{\deg((f)_n) \mid n < \omega\}$). For $I' \subseteq I$, I' is cofinal in I iff for every $a \in I$ there is a $b \in I'$ with $a \leq b$. f weakly represents (weakly subrepresents) I iff f represents (subrepresents) some I' cofinal in I. A degree **a** represents (subrepresents, weakly represents, weakly subrepresents) I iff some $f \in$ **a** does so. I is an ideal iff I is non-empty closed downward and under join.

Terminology. A tree T is a total function from $2^{<\omega} = \text{Str}$ into Str so that for any $\delta \in \text{Str}$, $T(\delta\hat{\ }0)$ and $T(\delta\hat{\ }1)$ are incompatible extensions of $T(\delta)$. $\delta \in \text{Str}(s)$ iff $\delta \in \text{Str}$ and $\text{dom}(\delta) = s$. A pre-tree of height s is a function T: $\text{Str}(s) \to \text{Str}$ where for all $\delta \in \text{Str}(s-1)$, $T(\delta\hat{\ }\langle 0 \rangle)$ and $T(\delta\hat{\ }\langle 1 \rangle)$ are incompatible extensions of $T(\delta)$. For $\delta \in \text{Str}$ and $A \in {}^\omega 2$, $\delta \subseteq A$ iff for all $i \in \text{dom}(\delta)$, $\delta(i) = A(i)$. Where T is a tree, $B \in [T]$ iff for some $A \in {}^\omega 2$; for all n, $T(A \upharpoonright n) \subset B$; (i.e. B is a path through T). Where T is a pre-tree of height s, $B \in [T]$ iff for some $\delta \in \text{Str}$, $\text{dom}(\delta) = s$ and $T(\delta) \subset B$.

Where T is a tree and $A \in {}^\omega 2$, let

$$\text{Code}(T, A)(\delta) = T(\langle A(0), \delta(0), \ldots, \delta(n-1), A(n) \rangle),$$

where $n = \text{dom}(\delta) - 1$. Notice: $\text{Code}(T, A)(\langle\,\rangle) \supsetneq T(\langle\,\rangle)$. Where T is a pre-tree of height $\leq 2n + 1$ and $\tau \in \text{Str}$, $\text{dom}(\tau) \geq n$, $\text{Code}(T, \tau)$ is defined similarly. For T a tree (pre-tree) and $B \in [T]$, let $\text{Coded}(B, T)$ be the real $A \in {}^\omega 2$ (string τ) such that $A(e) = i$ ($\tau(e) = i$) iff for some δ, $T(\delta) \subseteq B$ and $\delta(2e) = i$. If T is a pre-tree of height $2n$ or $2n + 1$,