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APPROPRIATE CROSS-SECTIONALLY SIMPLE FOUR-CELLS ARE FLAT

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When X is a set in E^n, we let $X_t = X \cap H_t$—where H_t is the horizontal hyperplane in E^n of height t. In this note, we prove that a 4-cell B in E^4, such that each nonempty slice B_t is either a point or a 3-cell, is flat whenever, for all t, B_t is flat in H_t and Bd B_t is flat in Bd B.

1. Introduction and summary. Throughout, we let H_t denote the horizontal hyperplane in E^n at height t, and when X is a set in E^n, we let $X_t = X \cap H_t$. In [**10**], it is proved that an $(n-1)$-sphere S in E^n $(n > 5)$ such that each nonempty slice S_t is either an $(n-2)$-sphere or a point has a 1-ULC complement whenever, for all t, S_t is flat in both H_t and S; subsequently, in [**9**] and [**11**] (see also [**17**]), $(n-1)$-spheres in E^n $(n > 4)$ with 1-ULC complements were shown to be flat. The necessity of these conditions is discussed in [**10**] and [**12**]. Similarly, a 2-sphere in E^3 such that each nonempty slice is a point or a 1-sphere was earlier shown to be flat in [**13**] and [**14**] with each relying upon the 1-ULC taming theorem of [**3**]. In this note, we extend this work to the case $n = 4$ by solving a similar question for a 4-cell; specifically, we prove the following:

THEOREM. *A* 4*-cell B in E^4, such that each nonempty slice B_t is either a point or a* 3*-cell, is flat whenever, for all t, B_t is flat in H_t and* Bd B_t *is flat in* Bd B.

The proof relies upon a condition—first described to us by R. J. Daverman in 1976—under which an n-cell in E^n is flat; Lemma 1 presents it. We include a proof because no reference contains the result; when $n > 4$, it is superceded by the 1-ULC taming theorems of [**3**], [**9**], and [**11**]; yet when $n = 4$, it has utility. (Daverman has pointed out that its hypotheses are strong enough to make the argument in Chernavskii [**7**] work too.)

LEMMA 1. *Let B be a* 4*-cell in E^4. If for each $\varepsilon > 0$ there exists an ε-self-homeomorphism h of E^4 supported in the ε-neighborhood of $E^4 - B$ such that $h(\text{Bd } B) \cap B = \varnothing$, then B is flat.*