PACIFIC JOURNAL OF MATHEMATICS
Vol. 108, No. 2, 1983

A NOTE ON PRIMARY POWERS OF A PRIME IDEAL

WEI-EIHN KUAN

Let $X \subset P_k^n$ be an irreducible projective variety. Let $B = k[x_0,\ldots,x_n]$ and let $P \subset B$ be the homogeneous prime ideal of X generated by $\operatorname{ht}(p)+1$ elements and let $A = B/P$ be the homogeneous coordinate ring of X. The following are equivalent: (1) $A_{(p)}$ is a complete intersection for all homogeneous prime ideals p in A of height 1; (2) P^2 is primary: (3) P^i is primary for all integers $i > 0$.

1. Introduction. In [**RV**, Theorem 3.3, p. 497], Robbiano and Valla proved the following: if $Y \subset X \subset P_k^n$ are projective schemes in the projective n-space over a field k, which are complete intersections in P_k^n and if Y is a positive dimensional irreducible, reduced normal subscheme of X with $(\operatorname{sing} X) \cap Y \subseteq \operatorname{sing} Y$, and if P is the prime ideal of Y in the homogeneous coordinate ring of X, then (a) P^2 is primary and (b) P^n is primary for every integer $n > 0$ if $\dim Y > \operatorname{codim} X$. [**RV**, Example 2, p. 560] gave an example of a projectively Gorenstein projective smooth curve in P_k^7 with homogeneous prime ideal P such that P^2 is not primary. It is known that an almost complete intersection is not Gorenstein. We prove the following theorem: Let X be a projective irreducible variety in P_k^n which is an almost complete intersection, i.e. the prime ideal P of X in the polynomial ring $k[x_0,\ldots,x_n]$ is generated by (codimen X) + 1 elements. Let $B = k[x_0,\ldots,x_n]$ and let $A = B/P = \oplus_{i\geq 0} A_i$ be the homogeneous coordinate ring of X, where A_i is the group of all homogeneous elements of degree i and $A_iA_j \subset A_{i+j}$. For a homogeneous ideal prime p in A let

$$A_{(p)} = \{a_i/s_i \mid a_i \in A_i,\ s_i \in A_i - p\}.$$

The following are equivalent:

(1) $A_{(p)}$ is a complete intersection for all homogeneous prime ideals p of height 1 in A.

(2) P^2 is primary.

(3) P^i is primary for all integers $i > 0$. Thus for an almost complete intersection X, "X is free of codim 1 singularities", is equivalent to "P^i is primary for all integers $i > 0$". Examples of projective varieties which are almost complete intersections are plentiful, for example, the Segre imbedding of $P_k^1 \times P_k^2$ into P_k^5 and the twist cubic in P_k^3 [**Sz**, p. 15–4]. The local case of (1) $\Leftrightarrow$ (2) was proved in [**K2**, Theorem, p. 1]: Let B be a regular