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ON THE VECTOR FIELDS ON AN ALGEBRAIC HOMOGENEOUS SPACE

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We construct a holomorphic vector field V with isolated zeros on an algebraic homogeneous space $X = G/P$ and show that the Koszul complex defined by V gives much information concerning the cohomology groups of X. Our results give useful examples to the studies of J. B. Carrell and D. Lieberman.

1. Koszul complex. Let X be a compact Kähler manifold of dimension n. We assume the manifold X admits a holomorphic vector field V whose zero set Z is simple isolated and nonempty. The following complex of sheaves is said to be the Koszul complex defined by V:

$$0 \to \Omega^n \xrightarrow{\partial} \Omega^{n-1} \xrightarrow{\partial} \cdots \to \Omega^1 \xrightarrow{\partial} \Omega^0 = \mathcal{O}_X \to 0, \tag{1.1}$$

where the differential ∂ is the contraction map $i(V)$. The structure sheaf of Z is $\mathcal{O}_Z = \mathcal{O}_X / i(V)\Omega^1$. To make the differentials of degree $+1$, we substitute $K^p = \Omega^{-p}$:

$$0 \to K^{-n} \xrightarrow{\partial} K^{-n+1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} K^0 \to 0. \tag{1.2}$$

For any locally free sheaf $\mathcal{F}$, we denote by $K(\mathcal{F})$ the complex obtained by tensoring $\mathcal{F}$ with (1.2) over $\mathcal{O}_X$. Let $\mathcal{H}^*(\mathcal{F})$ be the cohomology sheaves of the complex $K(\mathcal{F})$. Then, from the assumptions, it follows that $\mathcal{H}^q(\mathcal{F}) = 0$ for $-n \leq q < 0$ and $\mathcal{H}^0(\mathcal{F}) = \mathcal{F} \otimes \mathcal{O}_Z$, whose support is contained in Z. We abbreviate $\mathcal{F}_Z = \mathcal{F} \otimes \mathcal{O}_Z$. The hypercohomology $\mathbf{H}^*(X, K(\mathcal{F}))$ can be calculated by using the double Čech complex $\check{C}^*(\mathfrak{U}, K(\mathcal{F}))$ in the usual manner. See [3]. Corresponding to the natural two filtrations in $\check{C}^*(\mathfrak{U}, K(\mathcal{F}))$, we get the following spectral sequences which converge to $\mathbf{H}^{p+q}(X, K(\mathcal{F}))$:

$$'E_1^{p,q} = H^q(X, K^p(\mathcal{F})), \tag{1.3}$$

$$''E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{F})). \tag{1.4}$$

From the above remark, it follows that $\mathbf{H}^r(X, K(\mathcal{F})) = 0$ for $r \neq 0$ and $\mathbf{H}^0(X, K(\mathcal{F})) = H^0(Z, \mathcal{F}_Z)$. Note that the space $H^0(Z, \mathcal{O}_Z)$, i.e., in case $\mathcal{F} = \mathcal{O}_X$, can be interpreted as the ring of complex-valued functions on Z.