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A REMARK ON THE KASPAROV GROUPS $\mathrm{Ext}^i(A, B)$

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Let A and B be C^*-algebras. We show that, under reasonable assumptions (A unital, nuclear and separable, B with a strictly positive element), the groups $\mathrm{Ext}^i(A, B)$ of Kasparov are isomorphic—up to a shift of dimension—to the K-theory groups of some commutant of A in the outer multiplier algebra of $B \otimes \mathcal{K}$. The main tool to prove this is Kasparov's "generalized theorem of Voiculescu". Following an idea of Paschke, we use our result to get a part of the "generalized Pimsner-Voiculescu exact sequence" for crossed products.

0. Introduction. The purpose of this note is to provide a K-theoretic interpretation for the Kasparov groups $\mathrm{Ext}^0(A, B)$ and $\mathrm{Ext}^1(A, B)$ (where A, B are C^*-algebras), by realizing them as the K-groups of some commutant of A in the outer multiplier algebra $Q(B \otimes \mathcal{K})$ of $B \otimes \mathcal{K}$ (the relevant definitions are given in §1).

The role of commutants can be explained very simply. Let $\tau\colon A \to Q(B \otimes \mathcal{K})$ be an extension of A, and p be a projection in the commutant $\tau(A)'$ of $\tau(A)$; the mapping

$$\alpha_\tau(p)\colon A \to Q(B \otimes \mathcal{K})$$

defined by

$$\alpha_\tau(p)(a) = p \cdot \tau(a)$$

is still an extension of A. Similarly, if u is a unitary in $\tau(A)'$, the mapping

$$\alpha_\tau(u)\colon C_0(\mathbf{R}) \otimes A \to Q(B \otimes \mathcal{K})$$

defined by

$$\alpha_\tau(u)(f \otimes a) = f(u) \cdot \tau(a)$$

is an extension of $C_0(\mathbf{R}) \otimes A$. It is shown in Lemma 1 that these mappings α_τ extend to well-defined homomorphisms $K_i(\tau(A)') \to \mathrm{Ext}^{i+1}(A, B)$ ($i \in \mathbf{Z}/2$). Moreover, we show in Proposition 3 that under mild assumptions on A and B, it is possible to find an extension τ such that α_τ is actually an isomorphism.

1. Definitions and notations. Let A and B be C^*-algebras. In the study of extensions of the form

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0$$