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NORMS ON $F(X)$

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It is well known that if $\|\cdot\cdot\|$ is a norm on the field $F(X)$ of rational functions over a field F for which F is bounded, then $\|\cdot\cdot\|$ is equivalent to the supremum of a finite family of absolute values on $F(X)$, each of which is improper on F. Moreover, $\|\cdot\cdot\|$ is equivalent to an absolute value if and only if the completion of $F(X)$ for $\|\cdot\cdot\|$ is a field. We show that the analogous characterization of norms on $F(X)$ for which F is discrete is impossible by constructing for each infinite field F, a norm $\|\cdot\cdot\|$ on $F(X)$ such that F is discrete, $\|X\| < 1$, the completion of $F(X)$ for $\|\cdot\cdot\|$ is a field, but $\|\cdot\cdot\|$ is not equivalent to the supremum of finitely many absolute values.

1. Introduction and basic definitions. Let R be a ring and let $\mathfrak{T}$ be a ring topology on R, that is, $\mathfrak{T}$ is a topology on R making $(x, y) \to x - y$ and $(x, y) \to xy$ continuous from $R \times R$ to R. A subset A of R is *bounded* for $\mathfrak{T}$ if given any neighborhood U of zero, there exists a neighborhood V of zero such that $AV \subseteq U$ and $VA \subseteq U$. $\mathfrak{T}$ is a *locally bounded topology* on R if there exists a fundamental system of neighborhoods of zero for $\mathfrak{T}$ consisting of bounded sets.

We recall that a *norm* $\|\cdot\cdot\|$ on a ring R is a function from R to the nonnegative reals satisfying $\|x\| = 0$ if and only if $x = 0$, $\|x - y\| \leq \|x\| + \|y\|$ and $\|xy\| \leq \|x\|\,\|y\|$ for all x and y in R. If $\|\cdot\cdot\|$ is a norm on R, for each $\varepsilon > 0$ define B_ε by, $B_\varepsilon = \{r \in R: \|r\| < \varepsilon\}$. Then $\{B_\varepsilon: \varepsilon > 0\}$ is a fundamental system of neighborhoods of zero for a Hausdorff locally bounded topology $\mathfrak{T}_{\|\cdot\cdot\|}$ on R. Two norms on R are *equivalent* if they define the same topology. We note further that if $\|\cdot\cdot\|$ is a nontrivial norm on a field K (that is, $\mathfrak{T}_{\|\cdot\cdot\|}$ is nondiscrete), then a subset A of K is bounded for the topology defined by $\|\cdot\cdot\|$ if and only if A is bounded in norm.

It is classic that, to within equivalence, the only valuations on the field $F(X)$ of rational functions over a field F that are improper on F are the valuations v_p, where p is a prime polynomial of $F[X]$, and the valuation v_∞ defined by the prime polynomial X^{-1} of $F[X^{-1}]$ ([**1**, Corollary 2, p. 94]). For each valuation v, the function $|\cdot\cdot|_v$ defined by $|y|_v = 2^{-v(y)}$ for all y in $F(X)$ is an absolute value on $F(X)$ for which F is discrete. In [**2**, Theorem 2] we showed that if $\|\cdot\cdot\|$ is a nontrivial norm on $F(X)$ for which F is bounded, then $\|\cdot\cdot\|$ is equivalent to the supremum of finitely