PACIFIC JOURNAL OF MATHEMATICS
Vol. 109, No 2, 1983

SUBSYSTEMS OF THE POLYNOMIAL SYSTEM

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A pair of complex vector spaces (V, W) is a system if there is a C-bilinear map from $\mathbf{C}^2 \times V$ to W. Given any $\mathbf{C}[\zeta]$-module M, and (a, b) a fixed basis of $\mathbf{C}^2$, (M, M) is a system with $am = m$, $bm = \zeta m$ for all m in M. If $M = \mathbf{C}[\zeta]$, the system $P = (M, M)$ is called the polynomial system. The emphasis here is on the disparateness between the polynomial system and the polynomial module. It is shown that each nonzero formal power series in $\mathbf{C}[[\zeta]]$ determines a rank two subsystem of P. Among the consequences of this result are that:

(1) P contains c (c = cardinality of C) isomorphism classes of indecomposable subsystems of rank two.

(2) There is a complete set of invariants for decomposable extensions of $(0, \mathbf{C})$ by P.

It is also shown that extensions of finite-dimensional subsystems by P are isomorphic to subsystems of P. Consequently, P contains purely simple subsystems of arbitrary finite rank. Furthermore, a subsystem of P of finite rank is purely simple if and only if it is indecomposable. Finally the purely simple subsystems of P of rank two are shown to satisfy the ascending chain condition but not the descending chain condition.

Introduction. A pair of complex vector spaces (V, W) is a *system* if there is a $\mathbf{C}$-bilinear map from $\mathbf{C}^2 \times V$ to W. Any $\mathbf{C}[\zeta]$-module M ($\mathbf{C}[\zeta]$ is the ring of complex polynomials) gives rise to a system (M, M) with $am = m$, $bm = \zeta m$ where (a, b) is a fixed basis of $\mathbf{C}^2$. The category of systems contains, in this way, subcategories equivalent to the category of $\mathbf{C}[\zeta]$-modules. Probably the most significant difference between the theory of systems and that of modules over a principal ideal domain is the existence of purely simple systems of arbitrary finite rank. This paper is a step in the classification of such systems.

We begin with the simplest case: extensions (V, W) of finite-dimensional torsion-free systems by $P = (\mathbf{C}[\zeta], \mathbf{C}[\zeta])$. A formal power series $l = \sum_{k=0}^{\infty} \alpha_k \zeta^k$ may be regarded as a linear functional on $\mathbf{C}[\zeta]$, via $l(\zeta^k) = \alpha_k$. If $V = \mathbf{C}[\zeta]$, $W = V \oplus \mathbf{C}w$, $w \neq 0$, we make (V, W) into a system by setting $a\zeta^k = \zeta^k$, $b\zeta^k = \zeta^{k+1} + \alpha_k w$. This system, denoted by $(V, W)_l$, is an extension of $(0, \mathbf{C}w)$ by P. The rank of $(V, W)_l$ is 2, as seen in Theorem 3.1 of [**6**]. It is shown in Theorem 1.13 that any extension of a finite-dimensional indecomposable torsion-free system by P can be put in the above form. This is then used to show in Theorem 1.14 that any extension