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ON SYLVESTER'S PROBLEM AND HAAR SPACES

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Given a finite set of points in the plane (with distinct x coordinates) must there exist a polynomial of degree n that passes through exactly $n + 1$ of the points? Provided that the points do not all lie on the graph of a polynomial of degree n then the answer to this question is yes. This generalization of Sylvester's Problem (the $n = 1$ case) is established as a corollary to a version of Sylvester's Problem that holds for certain finite dimensional Haar spaces of continuous functions.

If E is a finite set of points in the plane then there exists a line through exactly two points of E unless all the points of E are colinear. This attractive result was posed as a problem by J. J. Sylvester in 1893 and was proved in 1933 by T. Gallai (see [**3**]). A particularly simple solution of Sylvester's Problem, due to L. M. Kelly, may be found in [**1**]. We ask the following question: If V_n is an n-dimensional vector space of real-valued continuous functions of a real variable and if E is a finite set in the plane, must there exist $g \in V_n$ so that the graph of g passes through exactly n points of E? We show that the answer to the above question is affirmative if V_n is a uni-modal Haar space of dimension n. (See Theorem 1.)

A Haar space H_n of dimension n on an interval $[a, b]$ is an n-dimensional real vector space of real-valued continuous functions with the additional property that if $g \in H_n$ and g has n distinct zeros then g is identically zero. Haar spaces are often also called Chebychev spaces. A Haar space H_n of dimension n is uni-modal if it satisfies the following: if $g \in H_n$ has $n - 1$ distinct zeros at $a \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} \leq b$ then g has a single change of monotonicity on each of the intervals

$$[\alpha_1, \alpha_2], [\alpha_2, \alpha_3], \dots, [\alpha_{n-2}, \alpha_{n-1}]$$

and g is monotonic on $[a, \alpha_1]$ and $[\alpha_{n-1}, b]$.

The algebraic polynomials of degree less than n form a uni-modal Haar space of dimension n on any interval. The following are other examples of uni-modal Haar spaces of dimension n on $[a, b]$:

(a) The space spanned by

$$\{1, e^{\alpha_1 x}, \dots, e^{\alpha_{n-1} x}\}$$

where $\alpha_1, \dots, \alpha_{n-1}$ are distinct non-zero real numbers.