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ENUMERATION OF SELF-DUAL CONFIGURATIONS

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A variety of combinatorial structures are self-dual in the sense that opposite elements have opposite properties. We provide a general enumeration theorem for these which has a number of interesting applications including the enumeration of self-dual boolean functions and 2-colorings of the vertices of polyhedra in which opposite vertices have different colors. Our method involves a modification of Pólya's enumeration theorem.

Introduction. Suppose that each of the twenty vertices of a dodecahedron is given one of two colors, say black or white. Then there are the following three enumeration problems that arise for these 2-colored configurations:

(1) What is the number of 2-colorings with an equal number of points of each color?

(2) What is the number of *self-complementary* 2-colorings, in which the configuration is unchanged on interchanging the colors?

(3) What is the number of *self-dual* 2-colorings, in which opposite points have different colors?

Note that we always regard two such colorings as equivalent if one can be brought into coincidence with the other by a rotation or reflection of the dodecahedron.

The answer to the first question, namely 1648, can be obtained by applying Pólya's enumeration theorem [**12**] to the cycle index of the automorphism group of the dodecahedron. The number of self-complementary colorings is 140 and this can be calculated using the approach of Read [**13**] in determining the number of self-complementary graphs. Finally, the number of self-dual colorings is 20, and this latter invariant can be computed using the modification of Pólya's theorem presented in this paper. Our approach is reminiscent of the enumeration of orientations of a graph [**5**, p. 128] where the appropriate permutation group is expressed in terms of two sets of variables. The resulting enumeration theorem can be used in a variety of interesting situations. We present applications to necklaces, polyhedra, and boolean functions.

Enumeration Theorem for Self-Dual Configurations

In this section we introduce our notation, provide a brief review of Pólya's enumeration theorem [**12**] and conclude with our modification for