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HOMOTOPICALLY TRIVIAL TOPOSES

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We give a number of equivalent conditions for a topos to be homotopically trivial and then relate these conditions to the logic of the topos. This is accomplished by constructing a family of intervals that can detect complemented, regular subobjects of the terminals. It follows that these conditions generally are weaker than the Stone condition but are equivalent to it if they hold locally. As a consequence we obtain an extension of Johnstone's list of conditions equivalent to DeMorgan's law. Thus, for example, the fact that there is no nontrivial homotopy theory in the category of sets is equivalent to the fact, among others, that maximal ideals in commutative rings are prime. Moreover, any topos has a 'best approximation' by a locally homotopically trivial topos.

1. Homotopy in a topos. The notion of (singular) homotopy in a topos is the notion of homotopy based, as in the topological case, on an interval, where by an *interval* I in a topos E is meant an internally linearly ordered object of E with disjoint minimum m: $1 \to I$ and maximum M: $1 \to I$ elements, i.e., $m \cap M = 0$. More precisely, for an interval I in E, an (ordered) pair of maps f, g: $A \to B$ in E is said to be *directly I-homotopic* (abbreviated DI-homotopic) if there is a map h: $A \times I \to B$ such that $f = h(\mathrm{id} \times m)$ and $g = h(\mathrm{id} \times M)$: $A \simeq A \times 1 \to A \times I \to B$, and to be *$I$-homotopic* if there is a finite sequence $\{j_k\}$, $k = 1,\dots,n+1$, of maps $A \to B$ with $j_1 = f$, $j_{n+1} = g$ and j_k DI-homotopic to j_{k+1} or vice versa, for $k = 1,\dots,n$. E is said to be *$(D)I$-homotopically trivial* if every pair of parallel maps in E are $(D)I$-homotopic. It is readily seen that E is both DI- and I-homotopically trivial for any interval I that is *trivial* in the sense that $I \simeq I_1 \amalg I_2$ and m, M factor through I_1, I_2 respectively. In fact we have:

1.1 PROPOSITION. *For an interval I in a topos E, the following conditions are equivalent*:

(1) *I is trivial.*

(2) *E is DI-homotopically trivial.*

(3) *E is I-homotopically trivial.*

We postpone the proof that (3) $\Rightarrow$ (1) until 2.2. Note that if I is trivial then DI-homotopy is both symmetric and transitive but the converse need