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ENDOSCOPIC GROUPS AND BASE CHANGE C/R

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We consider a real reductive group G with complex points $G(\mathbf{C})$, Galois automorphism σ, and real points $G(\mathbf{R}) = \{g \in G(\mathbf{C}): \sigma(g) = g\}$. In general, an irreducible admissible representation Π of $G(\mathbf{C})$ equivalent to its Galois conjugate $\Pi \circ \sigma$ need not be a lift from $G(\mathbf{R})$, even if G is quasi-split over $\mathbf{R}$. Following the results of L-indistinguishability we might expect this phenomenon to be related to the fact that σ-twisted conjugacy on $G(\mathbf{C})$ need not be "stable", and therefore attempt to match the various "unstable" combinations of σ-twisted orbital integrals on $G(\mathbf{C})$ with stable orbital integrals on certain groups $H(\mathbf{R})$. The principle of functoriality in the L-group would then suggest, with reservations in the nontempered case, a relation between the σ-twisted characters of representations of $G(\mathbf{C})$ fixed up to equivalence by σ and the "dual lifts" to $G(\mathbf{C})$ of stable characters on the groups $H(\mathbf{R})$.

In this paper we define the relevant groups H... they turn out to be the endoscopic groups from L-indistinguishability... and prove a matching theorem for orbital integrals. As a preliminary to the proposed dual liftings of characters we also study the "factoring" of Galois-invariant Langlands parameters for $G(\mathbf{C})$.

1. Introduction. We begin with two simple examples. Let $G(\mathbf{C}) = \mathbf{C}^x$ and $\sigma(z) = \bar{z}^{-1}$, $z \in \mathbf{C}^x$, so that $G(\mathbf{R}) = \{g \in G(\mathbf{C}): \sigma(g) = g\}$ is the unit circle in $\mathbf{C}^x$. A quasicharacter on $\mathbf{C}^x$ fixed by σ, i.e., trivial on the positive reals, need not be of the form $z \to \chi(z\sigma(z)) = \chi(z/\bar{z})$, with χ a character on the unit circle. At the same time $z \in \mathbf{C}^x$ is stably σ-conjugate to $-z$, but not σ-conjugate (see **[Sh6]** for definitions). Let $f \in C_c^\infty(\mathbf{C}^x)$ and write $f(r,\theta)$ for $f(re^{i\theta})$. Set $H_1 = H_2 = G$, so that $H_1(\mathbf{R}) = S^1$. Let

$$f_1(e^{i\theta}) = \frac{1}{2}\int_0^\infty (f(r,\theta/2) + f(r,\theta/2+\pi))\,dr/r$$

and

$$f_2(e^{i\theta}) = \frac{1}{2}e^{i\theta/2}\int_0^\infty (f(r,\theta/2) - f(r,\theta/2+\pi))\,dr/r$$

for $-\pi < \theta < \pi$. Then both f_1 and f_2 extend smoothly to S^1. If χ is a character on S^1 then $f \to \int_{-\pi}^{\pi} \chi(e^{i\theta}) f_1(e^{i\theta})\,d\theta$ is a distribution on $\mathbf{C}^x$ representing the usual lift of χ to $G(\mathbf{C})$, i.e., representing the quasicharacter $z \to \chi(z\sigma(z))$. On the other hand, $f \to \int_{-\pi}^{\pi} \chi(e^{i\theta}) f_2(e^{i\theta})\,d\theta$ lifts χ to the quasicharacter $z = re^{i\theta} \to \chi(z\sigma(z))e^{i\theta}$. We have therefore recovered the remaining Galois-invariant quasicharacters on $\mathbf{C}^x$.