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MINIMAL NONCOMMUTATIVE VARIETIES AND POWER VARIETIES

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A variety of finite monoids is a class of finite monoids closed under taking submonoids, quotients and finite direct products. A language L is a subset of a finitely generated free monoid. The variety theorem of Eilenberg sets up a one to one correspondence between varieties of finite monoids and classes of languages called, appropriately, varieties of languages. Recent work in variety theory has been concerned with relating operations on varieties of languages to operations on the corresponding variety of monoids and vice versa. For example, passing from a variety V of monoids to the variety PV generated by the power monoids of members of V corresponds to the operations of inverse substitution and literal morphism on varieties of languages. Recall that the power monoid of a monoid M is the power set PM with the usual multiplication of subsets. In this paper we consider iterating the operation which assigns PV to V. We show in particular that $\mathbf{P}^3\mathbf{V} = \mathbf{P}^4\mathbf{V}$ for any variety V and that the exponent 3 is the best possible. In fact if V contains a non-commutative monoid, then $\mathbf{P}^3\mathbf{V}$ is the variety of all finite monoids.

The proof of this theorem depends upon a classification of the minimal noncommutative varieties. A variety is minimal noncommutative if all its proper subvarieties contain only commutative monoids. We show that such a variety is either generated by a noncommutative metabelian group or by the syntactic monoid of one of the languages A^*a, aA^* or $\{ab\}$ over the alphabet $A = \{a, b\}$.

Let L be a language over the finite alphabet A. The syntactic monoid $M(L)$ of L is the quotient of the free monoid A^* by the largest congruence such that L is a union of classes. L is said to be recognizable if $M(L)$ is finite. Syntactic monoids have been used extensively to classify recognizable languages. For example, a language is rational if it can be obtained from the letters of the alphabet by applying the operations union, concatenation and star (or submonoid generated) a finite number of times. Kleene's theorem states that a language L is rational if and only if L is recognizable. A language is star-free if L can be obtained from the letters of A by applying the operations union, complement and concatenation a finite number of times. Schützenberger's theorem says that L is star-free if and only if $M(L)$ is a finite aperiodic monoid. That is every subgroup in $M(L)$ is trivial. These two important results are special cases of Eilenberg's variety theorem. We refer the reader to the books by Eilenberg [**1**] and Lallement [**2**] for details and many more examples.