PACIFIC JOURNAL OF MATHEMATICS
Vol. 111, No. 2, 1984

# $n$-GENERATOR IDEALS IN PRÜFER DOMAINS

RICHARD G. SWAN

**Heitmann has shown that a finitely generated ideal in a Prüfer domain of Krull dimension $n$ needs at most $n + 1$ generators. I will show here that this result is, in fact, the best possible. The same is true for Heitmann's stability theorems.**

These results, in particular, give additional counterexamples to the old question of whether a finitely generated ideal in a Prüfer domain can be generated by 2 elements [**GH**].[1] The first such example was given by Schülting [**S**] who found a 2-dimensional Prüfer domain with an ideal requiring 3 generators. His example is included in those discussed here, although the method of proof is very different.

A Prüfer domain may be characterized as a commutative integral domain in which every finitely generated ideal is invertible. Equivalently, it is a commutative integral domain $R$ such that the localization $R_P$ at any prime ideal is a valuation ring [**KC**, Th. 64]. A very thorough discussion of Prüfer rings is given in [**G**]. A noetherian Prüfer domain is a Dedekind ring, so it is natural to ask whether a Prüfer ring has properties similar to those of a Dedekind ring. The 2 generator question presumably first arose in this way. In some respects, a Prüfer ring behaves very much like a Dedekind ring. For example, every finitely generated torsion-free module is projective and a direct sum of ideals [**CE**, Ch. I, Prop. 6.1]. In addition, cancellation holds for such modules [**KA**, p. 75]. The results obtained here, however, show that in other aspects, a Prüfer ring behaves like a general commutative domain.

Let $\mu(M)$ denote the least number of generators of a module $M$.

THEOREM 1. *For any integer $n \geq 1$, there is a Prüfer domain $R$ of Krull dimension $n$ and an ideal $I_n$ of $R$ with $\mu(I_n) = n + 1$.*

THEOREM 2. *There is a Prüfer domain $R$ such that for every integer $n \geq 0$ there is an ideal $I_n$ of $R$ with $\mu(I_n) = n + 1$.*

---

[1] The question was first raised by Gilmer around 1964.