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A NUMBER THEORETIC SERIES OF I. KASARA

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The series

$$S(x) = 1 + \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{n_1 n_2 \cdots n_k \leq x \\ n_1, n_2, \ldots, n_k > 1}} \frac{1}{\log n_1 \log n_2 \cdots \log n_k}$$

is interpreted as a statement about Beurling generalized prime numbers and is estimated by means of Beurling theory.

This series was considered by I. Kasara in [**5**], in which he asserted that

$$\text{(1)} \qquad \text{“}S(x) = x + O(x/\log x).\text{”}$$

This assertion is not correct as it stands. We shall show that

$$\text{(2)} \qquad S(x) = cx + O\{x \exp(-(\log x)^{1/2-\varepsilon})\},$$

where $c \doteq 1.24292$.

We begin by giving the heuristic argument. Each integer in $(1, x]$ is uniquely expressible as a product of a certain number of primes. Thus we have

$$\text{(3)} \qquad [x] = 1 + \pi_1(x) + \pi_2(x) + \cdots$$

for $x \geq 1$, where

$$\pi_k(x) = \#\{n \leq x \colon n \text{ has exactly } k \text{ prime factors}\}$$

with repetitions allowed.

An estimate from prime number theory [**4**, §22.18] and a small calculation give, for each fixed k,

$$\text{(4)} \qquad \pi_k(x) \sim x(\log\log x)^{k-1}/\{(k-1)!\log x\}$$
$$\sim \frac{1}{k!} \sum_{\substack{n_1 n_2 \cdots n_k \leq x \\ n_1, n_2, \ldots, n_k > 1}} \frac{1}{\log n_1 \log n_2 \cdots \log n_k}.$$

This relation and (3) suggest formula (1). However, (4) does not hold uniformly in k, so this argument does not even show that $S(x) \sim cx$.