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HOMOLOGY OF COVERINGS

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This paper deals with an analysis of the first homology of a finite sheeted covering space of a complex and gives applications to some questions about 3-manifolds. Section 2 considers the relation between the property that a 3-manifold be virtually Haken and the, seemingly stronger, property that some finite sheeted cover has positive first betti number. Section 3 gives a procedure for computing the homology of a finite cover in terms of a presentation of the fundamental group of the base, and its action on the fiber and includes generalizations of the Fox-Goeritz theorem for cyclic covers to arbitrary abelian covers and to dihedral covers. Section 4 applies these theorems to 3-manifolds which have various types of symmetry and include some conditions which guarantee finite covers with positive first betti number. The paper concludes with a section of examples.

1. Introduction. The homology groups of the various covering spaces of a space M are of interest for a variety of reasons. For one thing, when restricted to some "characteristic" collection of coverings (the cyclic coverings of a knot space, for example) they provide invariants of the base which are much more tractable than its homotopy groups and, in general, much richer than its homology groups.

There is another point of particular interest to the study of 3-dimensional manifolds which we proceed to describe.

Following [**J**] we use the term *Haken manifold* to mean a compact, orientable, irreducible 3-manifold which is *sufficiently large* in the sense that it contains a 2-sided incompressible surface. The study of 3-manifolds splits nicely into the cases of finite fundamental groups and infinite fundamental groups. The vast majority of what is known in the second case — determination by homotopy type, existence of geometric structures, the homeomorphism and classification problems, etc. — is established only for Haken manifolds as can be discovered in the recent works of Jaco-Shalen [**JS**], Johannson [$\mathbf{J}_0$], Thurston [**T**], and the nice summary by Waldhausen [$\mathbf{W}_3$].

The examples of compact, orientable, irreducible 3-manifolds with infinite fundamental groups which are not Haken manifolds [$\mathbf{W}_1$], [**CJR**], [**FH**], [**HT**] seem to have the property that some finite sheeted cover is a Haken manifold. We will say that such a manifold is *virtually Haken*. This is consistent with the terminology from group theory which says that a