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HOLOMORPHIC FOLIATIONS AND DEFORMATIONS OF THE HOPF FOLIATION

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A deformation theory for transversally holomorphic foliations is developed here and used to give an explicit description of the transversally holomorphic foliations near the "Hopf foliations" on odd dimensional spheres.

Introduction. In [**1**] and [**2**] we began the study of the deformation theory of holomorphic foliations on a smooth compact manifold. Our aim was to construct a reasonably explicit parameterization of a neighborhood of a fixed holomorphic foliation $\mathcal{F}_0$ in the space of all foliations by generalizing Kuranishi's theorem on deformations of complex structures on compact complex manifolds. However, in [**1**] we assumed the existence of a smooth foliation $\mathcal{F}^{\perp}$ transverse to the foliation $\mathcal{F}_0$. The purpose of the present paper is to eliminate this rather artificial assumption. In [**3**] Gomez-Mont observed that the Kodaira-Spencer machine can be used to show the existence of such a parameterization by an analytic subset of a finite dimensional vector space. However, as is the case for the deformation theory of complex structures, the proof is rather abstract and is not easily adapted to computations. To illustrate our results, we present here a classification of all holomorphic foliations near the foliation given by the Hopf fibration $S^{2n+1} \to \mathbf{C}P^n$.

We shall now give a more precise statement of our results. The reader is assumed to be somewhat familiar with the notations and results of [**1**]; but we begin with a short review. Let $\mathcal{F}_0$ be a fixed holomorphic foliation of real codimensions $2q$ on the smooth, compact, oriented manifold M^n, i.e., $\mathcal{F}_0$ is given locally by smooth submersions into $\mathbf{C}^q$ which patch together via local biholomorphisms of $\mathbf{C}^q$. Let $L \subseteq TM$ and $Q = TM/L$ be the (real) tangent and normal bundles of $\mathcal{F}_0$ and fix once and for all a splitting $TM = L \oplus Q$ and a Riemannian metric on M respecting it. (In [**1**] this splitting was assumed to be induced by a transverse foliation. This is not necessarily the case here.) The complex structure map on Q induces a splitting of the complexified normal bundle in the standard way, $Q^{\mathbf{C}} = Q^{(1,0)} \oplus Q^{(0,1)}$ and there is a split exact sequence

$$(0.1) \qquad 0 \to E \underset{\overset{\curvearrowleft}{\tau}}{\overset{\iota}{\hookrightarrow}} TM^{\mathbf{C}} \underset{\overset{\curvearrowleft}{\rho}}{\overset{\pi}{\to}} Q^{(1,0)} \to 0$$