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GENERALIZED ORDERED SPACES WITH CAPACITIES

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We show that any *GO*-space having a capacity in the sense of Ščepin has a G_δ-diagonal and is perfect. In addition, such a space has a σ-discrete dense subset and a dense metrizable subspace, and any *GO*-space having a capacity and a point-countable base (or having a σ-discrete dense subset and a point-countable base) is metrizable.

1. Introduction. In [**14**] Ščepin defined a *capacity* for a space X to be a family of functions $\{\varepsilon_x \mid x \in X\}$ such that, for each closed $F \subset X$,

(C_1) $\varepsilon_x(F)$ is a non-negative real number with $\varepsilon_x(F) > 0$ iff $x \in \operatorname{Int}(F)$,

(C_2) if $F_1 \subset F_2$ are closed then $\varepsilon_x(F_1) \leq \varepsilon_x(F_2)$,

(C_3) for a fixed closed F, the function $x \to \varepsilon_x(F)$ is continuous,

(C_4) for a fixed x, if $\{F_\alpha \mid \alpha < \kappa\}$ is a family of closed sets satisfying $F_\alpha \supset F_\beta$ whenever $\alpha < \beta < \kappa$, then $\varepsilon_x(\bigcap_\alpha F_\alpha) = \inf_\alpha \varepsilon_x(F_\alpha)$.

In that same paper Ščepin announced without proof that a linearly ordered topological space (LOTS) having a capacity is metrizable. The purpose of this note is to prove a more general result from which Ščepin's result follows immediately, namely, that any *GO*-space (= suborderable space) with a capacity has a G_δ-diagonal. (Recall that the class of *GO*-spaces is precisely the class of subspaces of LOTS.) Along the way to that result, we show that any *GO*-space with a capacity is *perfect* (i.e., closed sets are G_δ). In §4 we will discuss two old questions about perfect *GO*-spaces in the context of *GO*-spaces having a capacity, proving that a *GO*-space with a capacity has a σ-discrete dense subset and a *GO*-space with a capacity and a point-countable base must be metrizable. Finally, examples in §5 show that our results are sharp.

Terminology and notation not defined in this paper will follow [**8, 11, 12**].

2. Preliminary results and perfect normality. We proceed via a sequence of lemmas.

2.1. LEMMA. *Any GO-space having a capacity is a first-countable space.*