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NONLINEAR ERGODIC THEOREMS FOR AN AMENABLE SEMIGROUP OF NONEXPANSIVE MAPPINGS IN A BANACH SPACE

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Let C be a nonempty closed convex subset of a Banach space, S a semigroup of nonexpansive mappings t of C into itself, and $F(S)$ the set of common fixed points of mappings t. Then we deal with the existence of a nonexpansive retraction P of C onto $F(S)$ such that $Pt = tP = P$ for each $t \in S$ and Px is contained in the closure of the convex hull of $\{tx: t \in S\}$ for each $x \in C$. That is, we prove nonlinear ergodic theorems for a semigroup of nonexpansive mappings in a Banach space.

1. Introduction. Let C be a nonempty closed convex subset of a real Banach space E. Then a mapping $T: C \to C$ is called nonexpansive on C if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T, that is,

$$F(T) = \{z \in C: Tz = z\}.$$

Let $S = \{S(t): t \geq 0\}$ be a family of nonexpansive mappings of C into itself such that $S(0) = I$, $S(t + s) = S(t)S(s)$ for all $t, s \in [0, \infty)$ and $S(t)x$ is continuous in $t \in [0, \infty)$ for each $x \in C$. Then S is said to be a nonexpansive semigroup on C.

The nonlinear ergodic theorem for nonexpansive mappings was originally studied in the framework of Hilbert spaces by Baillon [**1**], and later extended to Banach spaces by Bruck [**8**], Hirano [**15**], Reich [**21**] and others. A corresponding result for nonexpansive semigroups on C was given by Baillon [**2**], Baillon-Brézis [**3**] and Reich [**20**]. Nonlinear ergodic theorems for general commutative semigroups of nonexpansive mappings were given by Brézis-Browder [**4**] and Hirano-Takahashi [**16**]. Recentlly Takahashi [**26**] proved the following nonlinear ergodic theorem for a noncommutative semigroup of nonexpansive mappings: Let C be a nonempty closed convex subset of a real Hilbert space H, and let S be an amenable semigroup of nonexpansive mappings t of C into itself. Suppose

$$F(S) = \bigcap \{F(t): t \in S\} \neq \varnothing.$$