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TRANSFORMATION FORMULAE FOR MULTIPLE SERIES

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In this paper we prove a general transformation formula for a triple series of complex terms. We deduce a transformation formula for double series and discuss some applications. Furthermore, some reciprocity relations of the following type are obtained: Let a, b, c be reals greater than 1 and

$$P(a,b,c)=\sum_{r=1}^{\infty} r^{-a}\sum_{k=1}^{r} k^{-b}\sum_{l=1}^{k} l^{-c}.$$

Then

$$\begin{aligned}P(a,b,c)+P(a,c,b)+P(b,c,a)+P(b,a,c)&\\ +P(c,a,b)+P(c,b,a)&\\ =\zeta(a)\zeta(b)\zeta(c)+\zeta(a)\zeta(b+c)&\\ +\zeta(b)\zeta(c+a)+\zeta(c)\zeta(a+b)+2\zeta(a+b+c)&,\end{aligned}$$

where ζ denotes the Riemann zeta function. In particular, $P(2,2,2)=31\pi^6/15{,}120$.

1. Introduction. Since the time of Euler, the evaluation of certain infinite series, in closed form, in terms of the Riemann zeta function and allied functions is familiar. The processes involved, at times, yield recurrence relations among these functions. Results of this kind, dating back to 1743 and due to Euler, can be found in N. Nielsen's book (cf. [**6**], Erster Teil, Kapitel III). It appears that some recent authors are not aware of these results. For example, in 1953, G. T. Williams (cf. [**9**], Theorems III and I) proved the following results:

$$2\sum_{r=1}^{\infty}\frac{1}{r^a}\sum_{k=1}^{r}\frac{1}{k}=(a+2)\zeta(a+1)-\sum_{i=1}^{a-2}\zeta(a-i)\zeta(i+1), \tag{1.1}$$

$$\begin{aligned}\zeta(2)\zeta(2a-2)+\zeta(4)\zeta(2a-4)+\cdots+\zeta(2a-2)\zeta(2)&\\ =\left(a+\tfrac{1}{2}\right)\zeta(2a)&,\end{aligned} \tag{1.2}$$

where a is an integer ≥ 2 and ζ denotes the Riemann zeta function defined by $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$ for $s>1$. Williams claims that (1.1) is "apparently entirely new" and that the convolution considered in (1.2) "seems never to have been explicitly formulated before" (see also [**3**], [**4**], [**2**], and [**7**]). But we note that (1.1) and (1.2) are already deduced in