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THE MODULUS OF A DOUBLY CONNECTED REGION AND THE GEODESIC CURVATURE-AREA METHOD

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It is well known that the modulus of a doubly connected Riemann surface can be determined by the length-area method, that is, the method of extremal length, and that the extremal metric can be expressed in terms of a quadratic differential. Ahlfors introduced a related method based on the comparison of geodesic curvature and area. We show that the modulus of a doubly connected Riemann surface can be obtained by means of this geodesic curvature-area method. In the important special case in which there is a restriction on the curvature of the metrics, we identify all extremal metrics; they have constant curvature.

1. Introduction. A comparison of length and area has led to many important results in complex analysis. This method is based upon the fact that length and area are invariant under a conformal mapping when the metric undergoes a corresponding transformation. Ahlfors [**2**] considered a third conformally invariant quantity: geodesic curvature. He initiated a program based on the comparison of geodesic curvature and area. He presented the basic principles of the method and applied it to one simple case — the problem of estimating the conformal radius of a simply connected region. This is equivalent to estimating the hyperbolic metric on a simply connected region. He obtained explicit, sharp upper and lower bounds. The method is limited to smooth metrics. By making use of other methods, Minda [**5**] extended the upper bound to the class of $SK(k)$ metrics.

The work of Ahlfors indicates that the method has a wider range of applicability. But it is not clear that it will lead to equally explicit results in other situations. We apply the geodesic curvature-area method to the next simplest problem — estimation of the modulus of a doubly connected surface — and obtain an explicit, sharp upper bound.

2. Conformal metrics. In this section we gather together some basic facts concerning conformal metrics on Riemann surfaces. We adopt the convention, to be in effect for the remainder of the paper, that all metrics are positive and of class C^2. Let X be a Riemann surface and $\rho(z)|dz|$ a