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SPACES DETERMINED BY POINT-COUNTABLE COVERS

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Recall that a collection $\mathcal{P}$ of subsets of X is *point-countable* if every $x \in X$ is in at most countably many $P \in \mathcal{P}$. Such collections have been studied from several points of view. First, in characterizing various kinds of s-images of metric spaces, second, to construct conditions which imply that compact spaces and some of their generalizations are metrizable, and finally, in the context of meta-Lindelöf spaces. This paper will make some contributions to all of these areas.

1. Introduction. A space[1] X *is determined by* a cover $\mathcal{P}$, or $\mathcal{P}$ *determines* X,[2] if $U \subset X$ is open (closed) in X if and only if $U \cap P$ is relatively open (relatively closed) in P for every $P \in \mathcal{P}$. (For example, X is determined by $\mathcal{P}$ if $\mathcal{P}$ is locally finite or if $\{P^0: P \in \mathcal{P}\}$ covers X.) We will explore this notion in the context of point-countable covers.

Within the class of k-spaces, spaces determined by point-countable covers turn out to be related to spaces having a point-countable k-network. Recall that a cover $\mathcal{P}$ of X is a *k-network* for X if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$. Such collections have played a role in $\aleph_0$-spaces (i.e., regular spaces with a countable k-network [$\mathbf{M_1}$]) and $\aleph$-spaces (i.e., regular spaces with a σ-locally finite k-network [0]).[3]

Diagram I below gives a convenient overview of some of our principal results. The numbered conditions are defined as follows, where in (1.3) and elsewhere in this paper we adopt the convention that, if $\mathcal{C}$ is a collection of sets, then $\mathcal{C}^*$ denotes $\{\bigcup \mathcal{F}: \mathcal{F} \subset \mathcal{C}, \mathcal{F} \text{ finite}\}$.

(1.1) X has a point-countable cover $\mathcal{P}$ such that each open $U \subset X$ is determined by $\{P \in \mathcal{P}: P \subset U\}$.

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[1]All spaces are assumed to be Hausdorff.

[2]We use "X is determined by $\mathcal{P}$" instead of the usual "X has the weak topology with respect to $\mathcal{P}$". The term "weak topology" is used in a different way by functional analysts, to whom it implies that X has the smallest (not the largest) topology that gives each member of $\mathcal{P}$ its subspace topology. Our terminology avoids this problem, and is also shorter.

[3]While the elements of the defining k-network may always be chosen closed in $\aleph_0$-spaces and in $\aleph$-spaces (by simply taking closures), that is not always true for point-countable k-networks (see Example 9.2).